MODULI OF FORMAL TORSORS II

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Abstract. Applying the authors' preceding work, we construct a version of the moduli space of G-torsors over the formal punctured disk for a finite group G. To do so, we introduce two Grothendieck topologies, the sur (surjective) and luin (locally universally injective) topologies, and define P-schemes using them as variants of schemes. Our moduli space is defined as a P-scheme approximating the relevant moduli functor. We then prove that Fröhlich's module resolvent gives a locally constructible function on this moduli space, which implies that motivic integrals appearing the wild McKay correspondence are well-defined.

Contents

1. Introduction

In the preceding paper [10], the authors constructed the moduli stack of G-torsors over $\operatorname{Spec} k((t))$, where k is a field of characteristic $p > 0$ and G is a group of the form $H \rtimes C$ for a p-group H and a tame cyclic group C, which generalizes and refines Harbater's work for p-groups [7]. The motivation of the authors came from the wild McKay correspondence. In this theory, motivic integrals of the forms $\int_{\Delta_G} \mathbb{L}^{d-v}$ and $\int_{\Delta_G} \mathbb{L}^{w}$ appear, where Δ_G is the moduli space of G-torsors over $\operatorname{Spec} k((t))$, v, w are functions $\Delta_G \longrightarrow \frac{1}{|G|}\mathbb{Z}$ associated to a representation $G \longrightarrow \mathrm{GL}_d(k[[t]])$ and

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d is its rank. The first aim of the present paper is to construct a version of the moduli space Δ_G for an arbitrary finite group by using the mentioned result from the previous paper and prove that motivic integrals as above are well-defined in a version of the complete Grothendieck ring of varieties. After the first draft of this paper had been written, the main assertion of the wild McKay correspondence, the equality of $\int_{\Delta_G} \mathbb{L}^{d-v}$ and the stringy motive of the quotient $k[[t]]$-scheme $\mathbb{A}^d_{k[[t]]}/G$ associated to the representation, was proved by the second author in [18], building on the well-definedness of the integral obtained in this paper.

We do not construct the moduli stack, since it appears difficult. Instead we construct what we call the P-moduli space. This is a version of the moduli space, which is even coarser than the coarse moduli space. Actually this is the coarsest one for which motivic integrals as above still make sense. We construct the category of P-schemes by modifying morphisms of the category of schemes. The P-moduli space is the P-scheme approximating the relevant moduli functor the most. We call it the strong P-moduli space if it satisfies an additional condition. A precise statement of our first main result is as follows:

Theorem 1.1 (Theorem 8.9)**.** *Let G be a finite group and let k be a field. Consider the functor from the category of affine k-schemes to the category of sets which sends* $\operatorname{Spec} R$ *to the set of isomorphism classes of G-torsors over* $\operatorname{Spec} R((t))$*. This functor has a strong P-moduli space, which is the disjoint union of countably many affine schemes of finite type over k.*

The theorem can be generalized to the case where G is a finite étale group scheme (Corollary 8.10) and it holds in any characteristic. Here we outline the proof. From the previous work, we have the P-moduli space if G is the semidirect product of a p-group and a tame cyclic group. We construct the P-moduli space for an arbitrary G by "gluing" the P-moduli spaces of semidirect products as above. To do so, we show that every G-torsor over $\operatorname{Spec} R((t))$ is induced from an H-torsor with $H \subset G$ a subgroup which is a semidirect product as above, locally in $\operatorname{Spec} R$ for some Grothendieck topology. What we use as such a topology is the sur (surjective) topology; a scheme morphism $Y \longrightarrow X$ is a sur covering if it is surjective and locally of finite presentation. This topology is also incorporated into the very definition of P-schemes. We also introduce the luin (locally universally injective) topology. It is interesting that such a crude topology as the sur topology is still useful. The sur and luin topologies and P-schemes would be of independent interest and we study their basic properties. We note that Kelly [8, Def. 3.5.1] introduced a Grothendieck topology similar to the sur topology; he does not assume that a covering $Y \longrightarrow X$ is locally of finite presentation, instead assume that every point $x \in X$ admits a lift $y \in Y$ having the same residue field as x.

Advantages of P-moduli spaces are that it is much easier to show their existence than in the case of usual moduli stacks or schemes and that they are invariant by some transformations preserving geometric points. For instance if $\mathcal{F}$ is a moduli functor or stack and $f\colon \mathcal{F} \longrightarrow \frac{1}{l}\mathbb{Z}$ is a locally constructible function, then a P-moduli space for $\mathcal{F}$ is a disjoint union of P-moduli spaces for $\{f = r\} \subseteq \mathcal{F}$, the submoduli of $\mathcal{F}$ of objects where f has constant value r (see 4.27). For instance, we may restrict ourselves to those G-torsors over $\operatorname{Spec} R((t))$ which have constant ramification as a family over $\operatorname{Spec} R$ in a suitable sense.

We also prove that the functions v, w mentioned above are locally constructible. This together with Theorem 1.1 shows that integrals $\int_{\Delta_G} \mathbb{L}^{d-v}$ and $\int_{\Delta_G} \mathbb{L}^{w}$ are well-defined. The function v is essentially the same as the module resolvent introduced by Fröhlich [5] and w is a variant of v. When the given representation $G \longrightarrow \mathrm{GL}_d(k[[t]])$ is a permutation representation, then v and w are closely related to the Artin and Swan conductors [5, 15].

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2. Notation, terminology and convention

For a scheme X, we denote by $|X|$ the underlying topological space.

For a category $\mathbf{C}$, the expression $A \in \mathbf{C}$ means that A is an object of $\mathbf{C}$.

We denote the category of schemes by **Sch** and the one of affine schemes by **Aff**. For a scheme S, we denote the category of S-schemes by $\mathbf{Sch}/S$. When S is separated, we denote by $\mathbf{Aff}/S$ its subcategory of S-schemes affine over S.

We often identify a ring R with its spectrum $\operatorname{Spec} R$ and apply the terminology for schemes also to rings. For instance, for a ring map $A \longrightarrow B$ and a finite group G, we say that B is a G-torsor over A or that B/A is a G-torsor if $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ is a G-torsor.

3. Luin and sur topologies

In this section, we introduce two Grothendieck topologies, the luin topology and the sur topology, and study their basic properties. We need these topologies to develop the theory of P-schemes and P-moduli spaces in Section 4.

Definition 3.1. A morphism of schemes $f: Y \longrightarrow X$ is said to be *universally bijective* (resp. *universally injective*) if for all maps of schemes $X' \longrightarrow X$ the map $X' \times_X Y \longrightarrow X'$ is bijective (resp. injective) as map of sets.

Let S be a base scheme. A morphism of S-schemes $f: Y \longrightarrow X$ is said to be *geometrically bijective* (resp. *geometrically injective*, *geometrically surjective*) if for all algebraically closed field K and maps $\operatorname{Spec} K \longrightarrow S$ the map $\operatorname{Hom}_S(\operatorname{Spec}(K), Y) \longrightarrow \operatorname{Hom}_S(\operatorname{Spec}(K), X)$ is bijective (resp. injective, surjective).

Remark 3.2. A morphism of S-schemes $f: Y \longrightarrow X$ is geometrically bijective (resp. injective, surjective) if and only if it is so as a map of ($\operatorname{Spec}\mathbb{Z}$-)schemes. In other words this notion is an absolute property, not a relative one. We have chosen to include a base scheme to make the definition compatible with 4.8.

Lemma 3.3. *Let S be a base scheme and $f: Y \longrightarrow X$ be a morphism of S-schemes. We have:*

(1) the map f is universally injective if and only it is geometrically injective;
(2) if the map f is geometrically bijective (resp. geometrically surjective) then it is universally bijective (resp. surjective); the converse holds if f is locally of finite type.

Proof. (1) This is [9, Tag 01S4], taking into account that, if $K \longrightarrow L$ is a map of fields and Z is a scheme, then $Z(K) \longrightarrow Z(L)$ is injective.

(2) Since the base change of a surjective map is surjective, the "bijective case" follows from (1) and the "surjective" one. If a map if geometrically surjective then it

is clearly surjective. For the converse assume that f is locally of finite type and let $x\colon \operatorname{Spec} K \longrightarrow X \in X(K)$ be a geometric point. The fiber $Y \times_X \operatorname{Spec} K$ is locally of finite type, non empty since f is surjective, and has a K-rational point since K is algebraically closed. This defines an element of $Y(K)$ over $x \in X(K)$. □

Remark 3.4. Even if the notion of universally injective or bijective is more common in the literature, we are going to use the notion of geometrically injective or bijective instead. Firstly because, for morphisms locally of finite type, the two notions coincide. But the main reason is that the second notion easily extends in the case of natural transformation of functors (see 4.8) and it plays a crucial role in the next chapters.

Definition 3.5. A morphism of schemes $g\colon Y \longrightarrow X$ is called a *sur (surjective) covering* if it is locally of finite presentation and surjective.

A morphism of schemes $g\colon Y \longrightarrow X$ is called a *luin (locally universally injective) covering* if it is a sur covering and there is a covering $\{Y_i\}_i$ of open subsets of Y such that $Y_i \longrightarrow X$ is geometrically injective.

A morphism of schemes $g\colon Y \longrightarrow X$ is called a *ubi (universally bijective) covering* if it is a geometrically injective sur covering. In particular it is a geometrically bijective luin covering.

We call a collection of morphisms $(U_i \longrightarrow X)_{i\in I}$ in **Sch** a sur (resp. luin, ubi) covering if the induced morphism $g\colon Y = \coprod_{i\in I} U_i \longrightarrow X$ is a sur (resp. luin, ubi) covering.

It is easy to check that luin and sur coverings satisfy the axioms of a Grothendieck topology. We define the *luin topology* and the *sur topology* by these collections of coverings. By construction fppf coverings are sur coverings, while Zariski coverings are luin coverings.

If $(U_i \longrightarrow X)_{i\in I}$ is a sur (resp. luin) covering, then $\coprod_{i\in I} U_i \longrightarrow X$ is again a sur (resp. luin) covering. Hence, when discuss the luin or sur topology, we often consider coverings $U \longrightarrow X$ consisting of a single morphism.

We will soon prove (see 3.12) that sur coverings satisfy the equivalent conditions of [13, Proposition 2.33], that is for a sur covering any open affine below is dominated by a quasi-compact open above. This is the classical quasi-compactness condition required for fpqc coverings.

Definition 3.6 ([9, Tag 005G])**.** A subset $E \subseteq X$ of a topological space is called *constructible* if it is a finite union of sets of the form $U \cap (X - V)$ where $U, V \longrightarrow X$ are quasi-compact open immersions. It is said *locally constructible* if there exists an open covering $\{U_i\}_i$ of X such that $E \cap U_i$ is constructible in U_i.

Every constructible subset of a quasi-compact space is quasi-compact (see [9, Tag 09YH]). For a quasi-compact and quasi-separated scheme, locally constructible subsets are constructible (see [9, Tag 054E]). We will often use this form of Chevalley's theorem (see [9, Tag 054K]):

Theorem 3.7. *A quasi-compact and locally of finite presentation map of schemes preserves locally constructible subsets.*

Lemma 3.8. *Let $X = \operatorname{Spec} A$ be an affine scheme and $U, V \subseteq X$ two quasi-compact open subsets. Then there is a scheme structure on $E = U \cap (X - V)$ such that $E \longrightarrow X$ is a finitely presented immersion. If U is affine we can furthermore assume that E is affine.*

Proof. The quasi-compactness of V implies that there exists a finitely generated ideal I of A such that $\mathrm{Spec}\,(A/I) = X - V$. The composition $(\mathrm{Spec}\, A/I) \cap U \longrightarrow \mathrm{Spec}\, A/I \longrightarrow \mathrm{Spec}\, A$ is a finitely presented immersion whose image is $U \cap (X - V)$ and it is affine if U is affine. □

Lemma 3.9. *Let Y be a quasi-compact and quasi-separated scheme and $E \subseteq Y$ a constructible subset. If $E = \bigcup_{j\in J} E_j$ is union of constructible subsets of Y then there exists $J' \subseteq J$ finite such that $E = \bigcup_{j\in J'} E_j$.*

Proof. We use the theory of spectral spaces (see [9, Tag 08YF]). A quasi-compact and quasi-separated scheme is a spectral space. Any spectral space endowed with the coarsest topology in which its constructible subsets are both open and closed is quasi-compact. With respect to this topology, E is a closed subset of Y, hence quasi-compact and $E = \cup_{j\in J} E_j$ is an open covering. This implies the assertion. □

Corollary 3.10. *Let $\{f_j : Z_j \longrightarrow Y\}_{j\in J}$ be a collection of locally finitely presented and quasi-compact maps such that the image of $\coprod_{j\in J} Z_j \longrightarrow Y$ is locally constructible. If $V \subseteq Y$ is a quasi-compact and quasi-separated open subset of Y (e.g. affine) then there exists a finite subset $J' \subseteq J$ such that*

$$\mathrm{Im}(\coprod_{j\in J'} f_j^{-1}(V) \longrightarrow V) = \mathrm{Im}(\coprod_{j\in J} f_j^{-1}(V) \longrightarrow V).$$

Proof. If $E = \mathrm{Im}(\coprod_{j\in J} Z_j \longrightarrow Y)$ then the right hand side of the above equation is $E \cap V$, which is constructible in V. By Chevalley's theorem 3.7 the image E_i of $f_j^{-1}(V) \longrightarrow V$ is constructible. The conclusion follows from 3.9. □

Corollary 3.11. *Let $f : X \longrightarrow Y$ be a geometrically injective map which is locally of finite presentation. Then f is quasi-compact if and only if $f(X)$ is locally constructible (e.g. if f is geometrically bijective).*

Proof. The "only if" part is Chevalley's theorem 3.7, while the "if" part follows reducing first to the case where Y is affine and then applying 3.10 with $\{Z_j\}_j$ an open affine covering of X. □

Corollary 3.12. *Let $f : X \longrightarrow Y$ be a sur covering. Then for any quasi-compact open V of Y there exists a quasi-compact open W of X such that $f(W) = V$. In other words sur coverings satisfy the equivalent conditions stated in* [13, Proposition 2.33].

Proof. We can assume $V = Y$ affine. In this case it is enough to apply 3.10 to the collection $\{U \longrightarrow Y\}_U$ with U open affine of X. □

Definition 3.13. An open covering $\{U_i\}_{i\in I}$ of a topological space Y is called *locally finite* if for all $y \in Y$ there exists an open neighborhood U_y of y such that there are at most finitely many indices $i \in I$ with $U_y \cap U_i \neq \emptyset$. If all the U_i are quasi-compact (e.g. affine) this is the same of asking that for each $i \in I$ there are at most finitely many $j \in I$ with $U_i \cap U_j \neq \emptyset$.

Notation 3.14. Let Y be a set and let $\{Z_i\}_{i\in I}$ be a collection of subsets $Z_i \subset Y$. For a subset $J \subset I$, we define $Z_J^\circ := \bigcap_{i\in J} Z_i \setminus \bigcup_{i\in J^c} Z_i$.

It is easy to see that $Y = \coprod_{J \subset I} Z_J^\circ$. Moreover, for subsets $J_1, J_2 \subset I$, the set $Z_{J_1,J_2} := \bigcap_{i \in J_1} Z_i \setminus \bigcup_{i \in J_2} Z_i$ is written as

$$Z_{J_1,J_2} = \coprod_{J_1 \subset J,\, J_2 \subset J^c} Z_J^\circ. \tag{3.1}$$

Lemma 3.15. *Let Y be a scheme and $E \subseteq Y$ be a locally constructible subset. Then there exist affine schemes Z_j and locally finitely presented immersions $Z_j \longrightarrow Y$ such that the map $\coprod_j Z_j \longrightarrow Y$ has image E.*

If Y is quasi-separated and has a locally finite and affine open covering we can furthermore assume that the maps $Z_j \longrightarrow Y$ are quasi-compact and the map $\coprod_j Z_j \longrightarrow Y$ is a finitely presented monomorphism. In particular, Y has a ubi covering $\{Z_j \longrightarrow Y\}$ with Z_j affine.

Proof. We first prove the second assertion. Let Y be a quasi-separated scheme with a locally finite affine open covering $\{U_i\}_{i \in I}$. Consider the decomposition $Y = \coprod_{J \subset I} U_J^\circ$ as sets. From the local finiteness of the covering, if J is infinite, then U_J° is empty. Since $\{U_i\}$ is a covering, if J is empty, then so is U_J°. If J is finite and non empty and $j \in J$ then $U = \cap_{q \in J} U_q$ is a quasi-compact open subset of U_j because Y is quasi-separated. Since the covering $\{U_i\}_i$ is locally finite there are only finitely many $q \in J^c$ such that $U_q \cap U_j \neq \emptyset$ and therefore, using again that Y is quasi-separated, the union

$$V = \bigcup_{q \in J^c} (U_q \cap U_j)$$

is a quasi-compact open subset of U_j. Since $U_J^\circ = U \cap (U_j - V)$ by definition, Lemma 3.8 yields a structure of scheme on U_J° such that the morphism $U_J^\circ \longrightarrow U_j$ is a finitely presented immersion and, if Y is also separated so that U is affine, we can choose U_J° affine. In general U_J° is quasi-compact and separated and, since Y is quasi-separated, also the map $U_J^\circ \longrightarrow Y$ is a finitely presented immersion. Indeed the map $U_j \longrightarrow Y$ is a finitely presented immersion: it is locally of finite presentation because an open immersion, quasi-compact because Y is quasi-separated and U_j is quasi compact, and quasi-compact because it is a monomorphism. Moreover the map $\coprod_J U_J^\circ \longrightarrow Y$ is a surjective monomorphism.

We use the above construction several times. Firstly, starting from any locally finite affine open covering $\{U_i\}_{i \in I}$, it allows to reduce the problem to the case where Y is quasi-compact and separated: we can replace Y with U_J° and E with its preimage on U_J°.

In this case, since E is constructible, we can write $E = \bigcup_{l=1}^n V_l \setminus V_{n+l}$ with quasi-compact open subsets $V_l \subset Y$. We now apply the above constructions to a finite affine open covering $\{U_i\}_{i \in I}$ such that all V_l can be written as union of some opens in this covering. Since each $V_l \setminus V_{n+l}$ is a (automatically disjoint) union of subsets of the form U_J°, $J \subset I$ as in (3.1), so is E, say $E = \coprod_{J \in \Lambda} U_J^\circ$ for a set Λ of subsets of I. It follows that the map $\coprod_{J \in \Lambda} U_J^\circ \longrightarrow Y$ is finitely presented, a monomorphism, has image E and all $U_J^\circ \longrightarrow Y$ are finitely presented immersions. Moreover, since Y is separated, the U_J° are affine, as required.

For a general scheme Y, we take an affine covering $\{Y_i\}_i$ of Y and a finitely presented monomorphism $\coprod_j Z_{ij} \longrightarrow Y_i$ with image $E \cap Y_i$ and such that $Z_{ij} \longrightarrow Y_i$ is an affine and finitely presented immersion. It is clear that $\coprod_{i,j} Z_{ij} \longrightarrow Y$ satisfies the requests. $\square$

Corollary 3.16. *Let Y be a quasi-separated scheme with a locally finite and affine open covering. Then if $f\colon Z \longrightarrow Y$ is a luin covering there exists an ubi covering $Z' \longrightarrow Y$ which is finitely presented, separated and has a factorization $Z' \longrightarrow Z \xrightarrow{f} Y$. In particular a luin covering of Y is refined by an ubi covering.*

Proof. By 3.15, there exists a ubi covering $\{Y_j \longrightarrow Y\}$ with Y_j affine. For each j, we have the luin covering $Z_j := Z \times_Y Y_j \longrightarrow Y_j$. If $Z'_j \longrightarrow Y_j$ is a ubi covering as in the corollary for this luin covering, then $\coprod_j Z'_j \longrightarrow Y$ is the desired ubi covering. Thus it suffices to show the collary in the case where Y is affine.

By 3.12 there exists a quasi-compact open subset $\widetilde{Z} \subseteq Z$ such that $f(\widetilde{Z}) = Y$. In particular $\widetilde{Z} \longrightarrow Y$ is a luin covering refining the given one. Thus we can assume that Z is quasi-compact. Now let $\{Z_i\}_{i\in I}$ be a finite open covering by affine schemes of Z such that $Z_i \longrightarrow Y$ is geometrically injective and set $f(Z_i) = E_i$. Since f is quasi-compact and locally of finite presentation all the E_i are constructible subsets. We have $Y = \coprod_{\emptyset\neq J\subset I} E^\circ_J$ as sets. For each J, we choose an index $i_J \in J$ and let $Z_J \subset Z_{i_J}$ to be the preimage of E°_J, which is a constructible subset of Z_{i_J} mapping bijectively onto E°_J. Again from 3.15, there exists a finitely presented morphism $W_J \longrightarrow Z_{i_J}$ from an affine scheme W_J whose image is Z_J. The scheme $Z' = \coprod_J W_J$ with the map $Z' \longrightarrow Y$ satisfies the requests. □

4. P-schemes and moduli spaces

In this section, we develop the theory of P-schemes and P-moduli spaces. The category of P-varieties (P-schemes of finite type) can be regarded as the categorification of the modified Grothendieck ring of varieties (Definition 5.1). Although it would be more natural from this viewpoint to use the luin topology to define P-schemes, we actually use the sur topology. This is a key in later applications, since we have uniformization only locally in the sur topology (Section 7).

Notation 4.1. Starting from this section, we make use of Hom sets such as $\mathrm{Hom}_S(-,-)$, $\mathrm{Hom}_{\mathbf{Sch}/S}(-,-)$, $\mathrm{Hom}_{\mathbf{Aff}/S}(-,-)$ and we want to clarify here the notation. If $\mathcal{C}$ is a category and $X, Y\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ are two functor then $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ denotes the set of natural transformations $X \longrightarrow Y$. The symbols $\mathrm{Hom}_{\mathbf{Sch}/S}(-,-)$, $\mathrm{Hom}_{\mathbf{Aff}/S}(-,-)$ are used with this meaning for $\mathcal{C} = \mathbf{Sch}/S, \mathbf{Aff}/S$ respectively. On the other hand, by abuse of notation, we will often simply write $\mathrm{Hom}_S(-,-)$ if it is clear which category is used: $\mathrm{Hom}_S(X,Y)$ would be $\mathrm{Hom}_{\mathbf{Sch}/S}(X,Y)$ for $X, Y\colon (\mathbf{Sch}/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$, $\mathrm{Hom}_{\mathbf{Aff}/S}(X,Y)$ for $X, Y\colon (\mathbf{Aff}/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$.

If X and Y are S-schemes, then X, Y can be thought of as functors $h_X, h_Y\colon (\mathbf{Sch}/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ but also as their restriction $h^a_X, h^a_Y\colon (\mathbf{Aff}/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$. By Yoneda's lemma and Zariski descent we have canonical isomorphisms

$$\mathrm{Hom}_S(X,Y) \cong \mathrm{Hom}_{\mathbf{Sch}/S}(h_X, h_Y) \cong \mathrm{Hom}_{\mathbf{Aff}/S}(h^a_X, h^a_Y)$$

where $\mathrm{Hom}_S(X,Y)$ denotes the set of morphisms as S-schemes. We will simply write $\mathrm{Hom}_S(X,Y)$ and, depending on the interpretation of X, Y, use one of the above sets.

4.1. **P-morphisms and associated functor.** Let S be a base scheme. Let $\mathbf{Sch}/S$ (resp. $\mathbf{Aff}/S$) be the category of S-schemes (resp. affine schemes over S). By $\mathbf{Sch}'/S$ we denote either $\mathbf{Sch}/S$ or $\mathbf{Aff}/S$. As is well-known, associating the functor $T \longmapsto X(T)$ to the scheme X, we have a fully faithful embedding of $\mathbf{Sch}/S$ into the category of functors $(\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$. We often identify an S-scheme with the associated functor $(\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$.

Definition 4.2. We denote by $\mathbf{ACF}/S$ the category of algebraically closed fields K together with a map $\operatorname{Spec} K \longrightarrow S$. Given a functor $X\colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ (e.g. an S-scheme) we denote by X_F the restriction

$$X_F\colon \mathbf{ACF}/S \longrightarrow (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}.$$

Two elements $x\colon \operatorname{Spec} K \longrightarrow X$ and $y\colon \operatorname{Spec} K' \longrightarrow X$ of X_F are equivalent if there exists a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K'' & \longrightarrow & \operatorname{Spec} K' \\ \big\downarrow & & \big\downarrow{\scriptstyle y} \\ \operatorname{Spec} K & \xrightarrow{\ x\ } & X \end{array}$$

where K'' is a field. We denote by $|X|$ the set of equivalence classes of maps as above and we call points of X its elements. If $x \in X_F(K)$, with an abuse of notation, we will write $x \in |X|$. If $f\colon Y_F \longrightarrow X_F$ is a map of functors $(\mathbf{ACF}/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ we denote by $|f|$, or sometimes simply by f, the induced map $|Y| \longrightarrow |X|$.

Remark 4.3. Given a subset E of $|X|$ we define $E_F \subseteq X_F$ by

$$E_F(K) = \{s \in X(K) \mid s \in E \subseteq |X|\}.$$

Conversely given a subfunctor $G \subseteq X_F$ we can define the subset $|G| \subseteq |X|$ of points p for which there exists $x \in G(K)$ such that $x = p$ in $|X|$. Clearly $|E_F| = E$ for all $E \subseteq X$ and, in particular $|X_F| = |X|$. We also have $G \subseteq |G|_F$ with an equality if: for all $x \in X(K)$ and $K \longrightarrow K'$ if $x_{|K'} \in G(K')$ then $x \in G(K)$.

Notice that $(-)_F$ preserves fiber products of functors $(\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$. If X, Y are two schemes over S with Yoneda functors $h_{X/S}, h_{Y/S}\colon \mathbf{Sch}'/S \longrightarrow \mathbf{Set}$ then $h_{X/S} \times h_{Y/S}$ is the Yoneda functor of the S-scheme $X \times_S Y$, that is $h_{X \times_S Y/S} = h_{X/S} \times h_{Y/S}\colon \mathbf{Sch}'/S \longrightarrow \mathbf{Set}$. In particular we have $(X \times_S Y)_F = X_F \times Y_F$.

Definition 4.4. Given an S-scheme Y and a functor $X\colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$, a *$P$-morphism* $Y \longrightarrow X$ (over S) is a natural transformation $f\colon Y_F \longrightarrow X_F$ for which there exist a sur covering $\{g_i\colon Z_i \longrightarrow Y\}$ over S and morphisms $f_i'\colon Z_i \longrightarrow X$ over S making the following diagrams commutative

$$\begin{array}{ccc} (Z_i)_F & & \\ {\scriptstyle (g_i)_F}\big\downarrow & \searrow^{(f_i')_F} & \\ Y_F & \xrightarrow{\ f\ } & X_F \end{array} \tag{4.1}$$

We denote by $\operatorname{Hom}_S^{\mathrm{P}}(Y, X) \subseteq \operatorname{Hom}(Y_F, X_F)$ the set of P-morphisms from Y to X.

Since P-morphisms are stable by composition we define P-$\mathbf{Sch}/S$ as the category whose objects are S-schemes and whose maps are P-morphisms over S. An *S P-scheme* means an S scheme regarded as an object of P-$\mathbf{Sch}/S$.

If $X\colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a functor we define $X^{\mathrm{P}}\colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ as follows: $X^{\mathrm{P}}(Y) = \operatorname{Hom}_S^{\mathrm{P}}(Y, X) \subseteq \operatorname{Hom}(Y_F, X_F)$ is the set of P-morphisms $Y \longrightarrow X$.

There exists a natural functor $\mathbf{Sch}/S \longrightarrow P\text{-}\mathbf{Sch}/S$ sending an S-scheme to itself and a morphism to the induced P-morphism. Notice moreover that a P-morphism of schemes $Y \longrightarrow X$, more generally a functor $Y_F \longrightarrow X_F$, induces a map on the sets of points $|Y| \longrightarrow |X|$ which in general is not continuous.

Remark 4.5. We coined the terms, P-morphism and P-scheme, to connote "perfect" and piecewise. Indeed relative or absolute Frobenius maps of varieties in positive characterstic and, more generally, ubi coverings (for example given by locally closed decompositions) become isomorphisms as P-morphisms (see 4.14). In particular their inverses are examples of P-morphisms which are not necessarily morphisms of schemes. In the case of decompositions those morphisms do not even define continuous maps on the underlying topological spaces.

Proposition 4.6. *Let* $X\colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ *be a functor.*

1. *The functor* $X^{\mathrm{P}}\colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ *extends naturally to a functor* $(P\text{-}\mathbf{Sch}/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$.
2. *There is a canonical morphism* $X \longrightarrow X^{\mathrm{P}}$ *and* $X_F \longrightarrow (X^{\mathrm{P}})_F, |X| \longrightarrow |X^{\mathrm{P}}|$ *are isomorphisms. Moreover* $X^{\mathrm{P}} \longrightarrow (X^{\mathrm{P}})^{\mathrm{P}}$ *is an isomorphism.*
3. *Let* $Y \in \mathbf{Sch}'/S$, $f\colon Y \longrightarrow X$ *a P-morphism and* $\overline{f}\colon Y \longrightarrow X^{\mathrm{P}}$ *the corresponding element. Then* $Y_F \xrightarrow{f} X_F \cong (X^{\mathrm{P}})_F$ *coincides with* $\overline{f}_F\colon Y_F \longrightarrow (X^{\mathrm{P}})_F$.
4. *If* $Y\colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ *is another functor and using that* $Y_F \cong (Y^{\mathrm{P}})_F$ *we obtain a map*
$$\mathrm{Hom}_S(X, Y^{\mathrm{P}}) \longrightarrow \mathrm{Hom}_S(X_F, Y_F)$$
and this map is injective. If X *is an* S*-scheme its image is* $\mathrm{Hom}_S^{\mathrm{P}}(X,Y)$.
5. *If* $Y\colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ *is another functor then a map* $X \longrightarrow Y^{\mathrm{P}}$ *factors uniquely through a map* $X^{\mathrm{P}} \longrightarrow Y^{\mathrm{P}}$. *In other words the map* $X \longrightarrow X^{\mathrm{P}}$ *induces a bijection*
$$\mathrm{Hom}_S(X^{\mathrm{P}}, Y^{\mathrm{P}}) \longrightarrow \mathrm{Hom}_S(X, Y^{\mathrm{P}}).$$
In particular if X *is an* S*-scheme then*
$$\mathrm{Hom}_S(X^{\mathrm{P}}, Y^{\mathrm{P}}) \cong \mathrm{Hom}_S(X, Y^{\mathrm{P}}) \cong \mathrm{Hom}_S^{\mathrm{P}}(X,Y) \subseteq \mathrm{Hom}_S(X_F, Y_F).$$
6. *If* $Y\colon \mathbf{Sch}/S \longrightarrow \mathbf{Set}$ *is a sheaf in the Zariski topology and* X *is an* S*-scheme then*
$$\mathrm{Hom}_S^{\mathrm{P}}(X,Y) = \mathrm{Hom}_S^{P}(X, Y_{|\mathbf{Aff}/S}) \subseteq \mathrm{Hom}_S(X_F, Y_F).$$
In particular the restriction
$$\mathrm{Hom}_S(X, Y^{\mathrm{P}}) \longrightarrow \mathrm{Hom}_S(X_{|\mathbf{Aff}/S}, (Y_{|\mathbf{Aff}/S})^{\mathrm{P}})$$
is an isomorphism.
7. *If* U *is a reduced* S*-scheme and* X *is a scheme the map* $X(U) \longrightarrow X^{\mathrm{P}}(U)$ *is injective .*

Proof. 1) Consider the functor $\overline{X}\colon P\text{-}\mathbf{Sch}/S \longrightarrow \mathbf{Set}$ given by $\overline{X}(Y) = \mathrm{Hom}_{\mathbf{ACF}/S}(Y_F, X_F)$. The extension of X^{P} from $\mathbf{Sch}'/S$ to $P\text{-}\mathbf{Sch}/S$ is the subfunctor of $\overline{X}$ given by $Y \longmapsto \mathrm{Hom}_S^{\mathrm{P}}(Y,X)$.

2) If $T \in \mathbf{ACF}/S$ then the composite map
$$X(T) = X_F(T) \longrightarrow X^{\mathrm{P}}(T) = (X^{\mathrm{P}})_F(T) \hookrightarrow \mathrm{Hom}_{\mathbf{ACF}/S}(T_F, X_F)$$

is a bijection from the Yoneda lemma. This shows that the left map is a bijection and that $X_F \longrightarrow (X^{\mathrm{P}})_F$ is an isomorphism.

The last statement follows easily from the definitions.

3) By definition of P-morphism and X^{P} we can replace Y by a sur covering and assume that $f\colon Y_F \longrightarrow X_F$ is induced by a map $\hat{f}\colon Y \longrightarrow X$. In this case $Y \xrightarrow{\hat{f}} X \longrightarrow X^{\mathrm{P}}$ is exactly $\overline{f}$ and taking $(-)_F$ the conclusion follows.

4) In the first claim one can replace X by a scheme in $\mathbf{Sch}'/S$, in which case the result follows easily from 3). Assume now that X is an S-scheme and consider the last claim. If $X \in \mathbf{Sch}'/S$ the result is again clear by 3). If X is not necessarily in $\mathbf{Sch}'/S$, then $\mathrm{Hom}_S(X, Y^{\mathrm{P}})$ can be identified with the set of transformations $f\colon X_F \longrightarrow Y_F$ such that, for all $u\colon T \longrightarrow X$ with $T \in \mathbf{Sch}'/S$, the composition $f \circ u_F\colon T_F \longrightarrow Y_F$ is a P-morphism. Thus $\mathrm{Hom}^{\mathrm{P}}_S(X, Y) \subseteq \mathrm{Hom}_S(X, Y^{\mathrm{P}})$. Conversely if $f \in \mathrm{Hom}_S(X, Y^{\mathrm{P}})$, then it is Zariski locally a P-morphism and therefore sur locally induced by a genuine morphism. But this exactly means that f is a P-morphism, as required.

5) Given a map $\phi\colon X \longrightarrow Y^P$ it is easy to see that the unique extension $\phi^{\mathrm{P}}\colon X^{\mathrm{P}} \longrightarrow Y^{\mathrm{P}}$ is defined as follows. Given $a\colon U \longrightarrow X^{\mathrm{P}}$ one defines

$$\phi^{\mathrm{P}}(a)\colon U_F \xrightarrow{a_F} (X^{\mathrm{P}})_F \cong X_F \xrightarrow{\phi} (Y^{\mathrm{P}})_F \cong Y_F.$$

6) It is enough to note that from the sheaf condition on Y it follows that the map

$$\mathrm{Hom}_{\mathbf{Sch}/S}(Z, Y) \longrightarrow \mathrm{Hom}_{\mathbf{Aff}/S}(Z_{|\mathbf{Aff}/S}, Y_{|\mathbf{Aff}/S})$$

is an isomorphism for all S-schemes Z.

7) Consider $a, b\colon U \longrightarrow X$ such that $a_F = b_F$ and the Cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{h} & U \\ \downarrow & & \downarrow{\scriptstyle (a,b)} \\ X & \xrightarrow{\Delta} & X \times_S X \end{array}$$

Since $(-)_F$ preserves fiber products, it follows that $h_F\colon W_F \longrightarrow U_F$ is an isomorphism, that is $h\colon W \longrightarrow U$ is geometrically bijective. In particular $h(W) = U$ is closed as a subset of U. As h is an immersion it follows that it is a closed immersion (see [9, Tag 01IQ]) and therefore also an homeomorphism. Since U is reduced the map $h\colon W \longrightarrow U$ is an isomorphism, which means $a = b$. □

Lemma 4.7. *Let Y be a scheme over S and X a scheme locally of finite presentation and quasi-separated over S. Given a natural transformation $f\colon Y_F \longrightarrow X_F$ the following are equivalent:*

(1) there exist a luin covering $g\colon Z \longrightarrow Y$ over S and a map $f'\colon Z \longrightarrow X$ over S such that the diagram

$$\begin{array}{ccc} Z_F & & \\ {\scriptstyle g_F}\downarrow & \searrow^{f'_F} & \\ Y_F & \xrightarrow{f} & X_F \end{array} \tag{4.2}$$

is commutative;

(2) the transformation f is a P-morphism;

(3) if $\Gamma_f\colon Y_F \longrightarrow Y_F \times X_F = (Y \times_S X)_F$ *is the graph of* f *then* $C = \operatorname{Im}(|\Gamma_f|) \subseteq Y \times_S X$ *is locally constructible.*

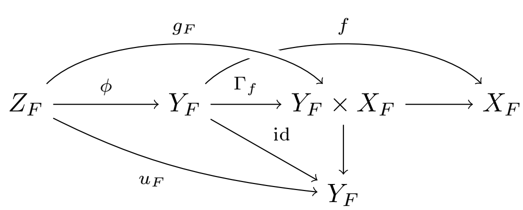
Proof. (1) $\Rightarrow$ (2) Follows by definition.

(2) $\Rightarrow$ (3) This is a local statement so that we can assume that Y is affine. Let $Z \xrightarrow{g} Y$ be a sur covering and $Z \xrightarrow{f'} X$ such that the diagram (4.2) is commutative. By 3.12 we can furthermore assume that $Z \longrightarrow Y$ is quasi-compact. The set $C = \operatorname{Im}(|\Gamma_f|)$ is the image of the graph $\Gamma_{f'}$ of f' along $Z \times_S X \longrightarrow Y \times_S X$. By Chevalley's theorem 3.7 it is therefore enough to show that $\Gamma_{f'}$ is locally constructible in $Z \times_S X$. But $X \longrightarrow S$ and therefore $Z \times_S X \longrightarrow Z$ are locally of finite presentation and quasi-separated. Therefore a section $Z \longrightarrow Z \times_S X$ is quasi-compact and locally of finite presentation. Chevalley's theorem 3.7 again shows that $\Gamma_{f'}$ is locally constructible.

(3) $\Rightarrow$ (1) By definition of luin coverings the statement is local in Y, so that we can assume Y affine. By 3.15 there are locally finitely presented immersions $Z_j \longrightarrow Y \times_S X$ from affine schemes such that $g\colon Z = \coprod_j Z_j \longrightarrow Y \times_S X$ has image C. The composition $u\colon Z \longrightarrow Y \times_S X \longrightarrow Y$ is locally of finite presentation because $X \longrightarrow S$ and hence $Y \times_S X \longrightarrow Y$ are locally of finite presentation. It is surjective because $C \longrightarrow Y$ is surjective. By definition it follows that $u\colon Z \longrightarrow Y$ is a sur covering. We are going to show that it is a luin covering, more precisely that $Z_j \longrightarrow Y$ is geometrically injective, and that $Z_F \xrightarrow{u_F} Y_F \xrightarrow{f} X_f$ is induced by the morphism $Z \longrightarrow Y \times_S X \longrightarrow X$, which will end the proof.

We first show that $C_F = \operatorname{Im}\Gamma_f \subseteq Y_F \times X_F$ (see 4.3). The inclusion $\operatorname{Im}\Gamma_f \subseteq C_F$ follows by construction. For the converse, let $K \in \mathbf{ACF}/S$ and $(y,x) \in C_F(K) \subseteq Y_F(K) \times X_F(K)$. Since $C = \operatorname{Im}|\Gamma_f|$, there exists $K \longrightarrow K'$ such that $(y,x)_{|K'} = \Gamma_f(\overline{y}) = (\overline{y}, f(\overline{y}))$ for some $\overline{y} \in Y_F(K')$. Thus $\overline{y} = y_{|K'}$ and $x_{|K'} = f(y_{|K'}) = f(y)_{|K'}$, which implies that $x = f(y)$ because $X(K) \longrightarrow X(K')$ is injective. In other words $\Gamma_f(y) = (y,x) \in \operatorname{Im}\Gamma_f$.

Since $\Gamma_f\colon Y_F \longrightarrow C_F$ is an isomorphism and the map $g_F\colon Z_F \longrightarrow (Y \times_S X)_F = Y_F \times X_F$ has image insides C_F, there is a factorization $g_F\colon Z_F \xrightarrow{\phi} Y_F \xrightarrow{\Gamma_f} Y_F \times X_F$. Moreover there is a commutative diagram



It follows that $\phi = u_F\colon Z_F \longrightarrow Y_F$ and that $f \circ \phi\colon Z_F \longrightarrow X_F$ is induced by $Z \longrightarrow Y \times_S X \longrightarrow X$. Since $Z_j \longrightarrow Y \times_S X$ is geometrically injective and $g_F = \Gamma_f \circ u_F\colon Z_F \longrightarrow (Y \times_S X)_F$ we can conclude that $Z_j \longrightarrow Z \xrightarrow{u} Y$ is geometrically injective, as required. □

Definition 4.8. A natural transformation $f\colon P \longrightarrow Q$ of functors $(\mathbf{ACF}/S)^{\mathrm{op}} \longrightarrow$ **Set** is said to be *geometrically bijective* (resp. *geometrically injective*, *geometrically surjective*) if it is an isomorphism (resp. injective, surjective).

A morphism $f\colon Y \longrightarrow X$ of functors $(\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow$ **Set** is said to be *geometrically bijective* (resp. *geometrically injective*, *geometrically surjective*) if so is

$f_F \colon Y_F \longrightarrow X_F$. Similarly a P-morphism $g \colon Z \longrightarrow X$ from an S-scheme is said to be *geometrically bijective* (resp. *geometrically injective*, *geometrically surjective*) if g, thought of as a map $Z_F \longrightarrow X_F$, is so.

Remark 4.9. The above definition extends the one given for morphisms of schemes (see 3.1).

If a morphism $f \colon Y \longrightarrow X$ of functors $(\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is geometrically bijective (resp. geometrically injective, geometrically surjective) then $|f| \colon |Y| \longrightarrow |X|$ is bijective (resp. injective, surjective), but the converse is not true.

Lemma 4.10. *Let $f \colon Y_F \longrightarrow X_F$ be a P-morphism of S-schemes. If f is an isomorphism in P-$\mathbf{Sch}/S$ then it is an isomorphism as natural transformation. The converse holds if Y and X are locally finitely presented and quasi-separated over S.*

Proof. The first statement is clear. For the second, by 4.7 we have that $C = \mathrm{Im}(|\Gamma_f|)$ is locally constructible in $Y \times_S X$. Since $\mathrm{Im}(|\Gamma_{f^{-1}}|)$ is the image of C via the automorphism $Y \times_S X \cong X \times_S Y$, it follows that $\mathrm{Im}(|\Gamma_{f^{-1}}|)$ is locally constructible and therefore $f^{-1} \colon X_F \longrightarrow Y_F$ is a P-morphism by 4.7. □

Lemma 4.11. *Let $X \colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a functor.*

(1) The functor $X^{\mathrm{P}} \colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a sheaf in the sur topology.
(2) If the diagonal of X is representable and of finite presentation then $X \longrightarrow X^P$ is a sheafification morphism for the sur topology.
(3) If X is a scheme locally of finite presentation and quasi-separated over S then $X \longrightarrow X^{\mathrm{P}}$ is a sheafification morphism for the luin topology.

Proof. 1) Consider $\overline{X} \colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ defined by $\overline{X}(Y) = \mathrm{Hom}(Y_F, X_F)$. Since $(-)_F$ preserves fiber products and, if $V \longrightarrow U$ is a sur covering, the maps $V_F(K) \longrightarrow U_F(K)$ are surjective for all algebraically closed fields, it is easy to see that $\overline{X}$ is a sheaf in the sur topology. Moreover $X^{\mathrm{P}} \subseteq \overline{X}$ is a subfunctor. By definition of X^{P} we see that if an object of $\overline{X}$ is sur locally in X^{P} then it belongs to X^{P}. This implies that X^{P} is a subsheaf of $\overline{X}$.

2) The map $X \longrightarrow X^{\mathrm{P}}$ is by definition an epimorphism in the sur topology. We need to check that if $a, b \in X(U)$ become equal in $X^{\mathrm{P}}(U)$ then they are sur locally equal. In particular we can assume U affine. Let $W \longrightarrow U$ be the base change of the diagonal $X \longrightarrow X \times_S X$ along the map $(a, b) \colon U \longrightarrow X \times_S X$. By hypothesis W is an algebraic space and $W \longrightarrow U$ is of finite presentation. Moreover a, b become equal in $X(W)$. Since $a_F = b_F \colon U_F \longrightarrow X_F$ we can conclude that $W_F \longrightarrow U_F$ is bijective. Since W is an algebraic space there exists an étale atlas $V \longrightarrow W$ from a scheme. The resulting map $V \longrightarrow U$ is locally of finite presentation, surjective and therefore a sur covering.

3) Now assume that X is a scheme locally of finite presentation and quasi-separated over S. By 4.7 it follows that $X \longrightarrow X^{\mathrm{P}}$ is an epimorphism in the luin topology. As before we need to check that if $a, b \in X(U)$ become equal in $X^{\mathrm{P}}(U)$ then they are luin locally equal. By the same argument above we see that they are equal after a map $W \longrightarrow U$ which is locally of finite presentation, quasi-compact and geometrically bijective. The difference now is that W is a scheme and therefore $W \longrightarrow U$ is a luin covering. □

Corollary 4.12. *Let* $\mathrm{Sh}_{\mathrm{sur}}(\mathbf{Sch}'/S)$ *be the category of sur sheaves on* $\mathbf{Sch}'/S$. *The functor* $(-)^{\mathrm{P}}$ *determines a fully faithful functor*

$$P\text{-}\mathbf{Sch}/S \longrightarrow \mathrm{Sh}_{\mathrm{sur}}(\mathbf{Sch}'/S)$$

Proof. For an S-scheme X, the functor X^P is a sur sheaf from Lemma 4.11. From Proposition 4.6 (5), for S-schemes X and Y, we have a natural bijection

$$\mathrm{Hom}_S(X^P, Y^P) \longrightarrow \mathrm{Hom}_S^P(X, Y),$$

which proves the corollary. □

4.2. **P-moduli spaces.**

Definition 4.13. Let $\mathcal{F}\colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a functor. A *P-moduli space* of $\mathcal{F}$ is an S-scheme X together with a morphism $\pi\colon \mathcal{F} \longrightarrow X^{\mathrm{P}}$ such that

(1) π is geometrically bijective, that is the induced map $\pi_F\colon \mathcal{F}_F \longrightarrow (X^{\mathrm{P}})_F$ is an isomorphism;
(2) π is universal, that is for any morphism $g : \mathcal{F} \longrightarrow Y^{\mathrm{P}}$ where Y is a scheme over S, there exists a unique S-morphism $f : X^{\mathrm{P}} \longrightarrow Y^{\mathrm{P}}$ with $f \circ \pi = g$.

If this is the case, we also call the morphism π a *P-moduli space*. It is clear that if exists, a P-moduli space is unique up to unique P-isomorphism.

Lemma 4.14. *Let* $\phi\colon \mathcal{F} \longrightarrow \mathcal{G}$ *be a map of functors* $(\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$. *If* ϕ *is geometrically bijective then* $\phi^{\mathrm{P}}\colon \mathcal{F}^{\mathrm{P}} \longrightarrow \mathcal{G}^{\mathrm{P}}$ *is a monomorphism. If* ϕ *is also an epimorphism in the sur topology then* $\phi^{\mathrm{P}}\colon \mathcal{F}^{\mathrm{P}} \longrightarrow \mathcal{G}^{\mathrm{P}}$ *is an isomorphism.*

In particular a locally finitely presented and geometrically bijective morphism of S*-schemes, that is an ubi covering, is a* P*-isomorphism.*

Proof. The first claim follows from 4.6, 4) and 5). For the second one it is enough to recall that $\mathcal{F}^{\mathrm{P}}$ and $\mathcal{G}^{\mathrm{P}}$ are sheaves in the sur topology. □

Remark 4.15. Let $\pi\colon \mathcal{F} \longrightarrow X^{\mathrm{P}}$ be a geometrically bijective map, so that, by 4.14, $\pi^{\mathrm{P}}\colon \mathcal{F}^{\mathrm{P}} \longrightarrow X^{\mathrm{P}}$ is a monomorphism. Then π is a P-moduli space if and only if for all maps $\mathcal{F} \longrightarrow Y^{\mathrm{P}}$ the map $(X^{\mathrm{P}})_F \cong \mathcal{F}_F \longrightarrow (Y^{\mathrm{P}})_F$ is a P-morphism $X \longrightarrow Y$.

In particular $\mathcal{F} \longrightarrow X^{\mathrm{P}}$ is a P-moduli space if and only if $\mathcal{F}^{\mathrm{P}} \longrightarrow X^{\mathrm{P}}$ is a P-moduli space and vice versa.

Definition 4.16. Let $\mathcal{F}\colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a functor. A *strong P-moduli space* for $\mathcal{F}$ is an S-scheme X together with a morphism $\pi\colon \mathcal{F} \longrightarrow X^{\mathrm{P}}$ such that $\pi^{\mathrm{P}}\colon \mathcal{F}^{\mathrm{P}} \longrightarrow X^{\mathrm{P}}$ is an isomorphism.

A strong P-moduli space is also unique up to unique P-isomorphism. From Remark 4.15 it follows that a strong P-moduli space is a P-moduli space.

Proposition 4.17. *Let* $\mathcal{F}\colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ *be a functor,* X *an* S*-scheme and* $\pi\colon \mathcal{F} \longrightarrow X^{\mathrm{P}}$ *be a morphism. Then* π *is an epimorphism in the sur topology if and only if there exist a sur covering* $\{Z_i \xrightarrow{g_i} X\}$ *and commutative diagrams*

$$\begin{array}{ccc} Z_i & \xrightarrow{g_i} & X \\ \downarrow & & \downarrow \\ \mathcal{F} & \xrightarrow{\pi} & X^{\mathrm{P}} \end{array}$$

The map $\pi\colon \mathcal{F} \longrightarrow X^{\mathrm{P}}$ *is a strong P-moduli space if and only if it is geometrically bijective and an epimorphism in the sur topology.*

Proof. The first statement follows from the fact that $X \longrightarrow X^{\mathrm{P}}$ is an epimorphism in the sur topology. The second one from 4.14. □

Remark 4.18. From the point of view of moduli theory a more natural definition of P-moduli space would have been to admits S-algebraic spaces in the above definitions. Since a quasi-separated algebraic space has a dense open subset which is a scheme, it follows that for a finite dimensional quasi-separated algebraic space Y there exists a geometrically bijective map $X_1 \amalg \cdots \amalg X_n \longrightarrow Y$ in which all X_i are schemes and all maps $X_i \longrightarrow Y$ are immersions. In particular such a Y always has a strong P-moduli space.

So in concrete cases there is no need to use algebraic spaces and also this let us avoid to deal with locally constructible subsets of algebraic spaces.

Definition 4.19. By a *geometric property* $\mathcal{Q}$ for a functor $\mathcal{F} \colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ we mean a subset $\mathcal{Q} \subseteq |\mathcal{F}|$. By 4.3 this can be thought of as a subfunctor $\mathcal{Q}$ of $\mathcal{F}_F$ with the following property: for all maps $a \colon K \longrightarrow K'$ in $\mathbf{ACF}/S$ and all $x \in \mathcal{F}(K)$, if x is mapped by $\mathcal{F}(K) \longrightarrow \mathcal{F}(K')$ to an element of $\mathcal{Q}(K')$ then $x \in \mathcal{Q}(K)$. Given a geometric property $\mathcal{Q}$ of $\mathcal{F}$ we define the subpresheaf

$$\mathcal{F}^{\mathcal{Q}}(V) = \{V \xrightarrow{A} \mathcal{F} \mid A(V) \subseteq \mathcal{Q} \subseteq |\mathcal{F}|\}.$$

A *locally constructible* property for $\mathcal{F}$ is a geometric property $\mathcal{Q}$ for $\mathcal{F}$ satisfying the following condition: for every S-scheme V and map $A \colon V \longrightarrow \mathcal{F}$ the inverse image $A^{-1}(\mathcal{Q}) \subseteq V$ is a locally constructible subset of V.

Remark 4.20. If X is an S-scheme a geometric property (resp. locally constructible property) of X is a subset (resp. a locally constructible subset) of X.

Proposition 4.21. *Let X be an S-scheme and $\mathcal{Q}$ be a locally constructible subset of X. Let also $Q = \coprod_i Q_i \longrightarrow X$ be a geometrically injective map with image $\mathcal{Q}$ and where the maps $Q_i \longrightarrow X$ are finitely presented immersions. Then the map $Q \longrightarrow X^{\mathcal{Q}}$ induces an isomorphism $Q^{\mathrm{P}} \cong (X^{\mathcal{Q}})^{\mathrm{P}}$.*

Proof. Since the image of $Q \longrightarrow X$ is in $\mathcal{Q}$ the map $Q \longrightarrow X^{\mathcal{Q}}$ is geometrically bijective. Moreover if V is a scheme with a map $V \longrightarrow X^{\mathcal{Q}}$, that is a map $V \longrightarrow X$ with image in $\mathcal{Q}$, then $Q \times_{X^{\mathcal{Q}}} V = Q \times_X V \longrightarrow V$ is geometrically bijective, locally of finite presentation and therefore, by 3.11, a luin covering. By 4.14 we get the result. □

Remark 4.22. If $\mathcal{F} \colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a functor then $\mathcal{F}$ and $\mathcal{F}^{\mathrm{P}}$ have the same geometric properties since $|\mathcal{F}| = |\mathcal{F}^{\mathrm{P}}|$. Moreover if $\mathcal{Q}$ is such a property then the inclusion $\mathcal{F}^{\mathcal{Q}} \longrightarrow \mathcal{F}$ induces an isomorphism $(\mathcal{F}^{\mathcal{Q}})^{\mathrm{P}} \longrightarrow (\mathcal{F}^{\mathrm{P}})^{\mathcal{Q}}$.

Lemma 4.23. *Let $\mathcal{F}, \mathcal{G} \colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be functors, $\mathcal{Q}$ be a geometric property for $\mathcal{G}$ and $\phi \colon \mathcal{F}^{\mathrm{P}} \longrightarrow \mathcal{G}^{\mathrm{P}}$ be a sur epimorphism. If $\phi^{-1}(\mathcal{Q}) \subseteq |\mathcal{F}|$ is locally constructible for $\mathcal{F}$ then $\mathcal{Q}$ is locally constructible for $\mathcal{G}$. In particular $\mathcal{F}$ and $\mathcal{F}^{\mathrm{P}}$ have the same locally constructible properties.*

Proof. Let $V \xrightarrow{A} \mathcal{G}$. We have to show that $A^{-1}(\mathcal{Q}) \subseteq V$ is locally constructible. In particular we can assume that V is affine. Since ϕ is a sur epimorphism, shrinking

V more if necessary, there is a commutative diagram

$$\begin{array}{ccccc} W & \xrightarrow{B} & \mathcal{F} & \longrightarrow & \mathcal{F}^{\mathrm{P}} \\ \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle \phi} \\ V & \xrightarrow{A} & \mathcal{G} & \longrightarrow & \mathcal{G}^{\mathrm{P}} \end{array}$$

where $g\colon W \longrightarrow V$ is a sur covering of schemes. By 3.12 we can assume that W is quasi-compact and hence g is quasi-compact. As g is surjective we obtain

$$g(B^{-1}(\phi^{-1}(\mathcal{Q}))) = A^{-1}(\mathcal{Q})$$

which is then locally constructible thanks to Chevalley's theorem 3.7. □

Proposition 4.24. *Let $\mathcal{F}\colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a functor and $\mathcal{Q}$ be a locally constructible property for $\mathcal{F}$. If $\mathcal{F}$ has a strong P-moduli space X which is quasi-separated and admits a locally finite and affine open covering then $\mathcal{F}^{\mathcal{Q}}$ has a strong P-moduli space Y which is a disjoint union of affine schemes. If moreover X is locally of finite presentation over S so is Y.*

Proof. By 4.22 and 4.23 we can assume that $\mathcal{F} = X$. The result follows from 3.15 and 4.21. □

Definition 4.25. If I is a set and $\mathcal{F}\colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a functor, a map (or function) $g\colon \mathcal{F} \longrightarrow I$ is just a map $g\colon |\mathcal{F}| \longrightarrow I$. The map g is called *locally constructible* if $g^{-1}(i) \subseteq |\mathcal{F}|$ is locally constructible for all $i \in I$.

Proposition 4.26. *Let $\phi\colon \mathcal{G} \longrightarrow \mathcal{F}$ be a morphism of functors $(\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ and $g\colon \mathcal{F} \longrightarrow I$ be a map. If g is locally constructible then so is $g \circ \phi$. The converse holds if ϕ is a sur covering. In particular $\mathcal{F}$ and $\mathcal{F}^{\mathrm{P}}$ have the same locally constructible functions.*

Proof. The first claim follows from definition. For the converse, let $A\colon V \longrightarrow \mathcal{F}$ be a map from an S-scheme. We need to show that $g \circ A\colon V \longrightarrow I$ is locally constructible. By hypothesis there exists a sur covering $\{\phi_j\colon V_j \longrightarrow V\}$ such that all $g \circ A \circ \phi_i$ are locally constructible. Since being a locally constructible subset is a Zariski local property, we can assume that the sur covering has only one element and that V is affine. In other words we can assume that $\mathcal{F} = V$, $\mathcal{G} = W$ is a scheme, $\phi\colon W \longrightarrow V$ is a sur covering and, by 3.12, that W is quasi-compact. As $\phi((g \circ \phi)^{-1}(i)) = g^{-1}(i)$ for all i, the conclusion follows from Chevalley's theorem 3.7. □

Proposition 4.27. *Let $\mathcal{F}\colon (\mathbf{Sch}'/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a functor and $g\colon \mathcal{F} \longrightarrow I$ be a locally constructible function. Then the maps $\mathcal{F}^{g^{-1}(i)} \longrightarrow \mathcal{F}$ induce a map*

$$\coprod_{i \in I} (\mathcal{F}^{g^{-1}(i)})^{\mathrm{P}} \longrightarrow \mathcal{F}^{\mathrm{P}}$$

where $\coprod$ is the union as Zariski sheaves, which is geometrically bijective and a luin epimorphism. If X_i is a strong P-moduli space for $\mathcal{F}^{g^{-1}(i)}$ then $\coprod_i X_i$ is a strong P-moduli space for $\mathcal{F}$.

Proof. The map in the statement is well defined because $\mathcal{F}^{\mathrm{P}}$ is a Zariski sheaf by 4.11 and it is clearly geometrically bijective. We now show that it is an epimorphism. Consider $A\colon V \longrightarrow \mathcal{F}$ and set $g_A = A \circ g\colon V \longrightarrow I$. We can assume that V is an affine scheme. By 3.15 for all $i \in I$ there exists a finitely presented monomorphism $W_i \longrightarrow V$ whose image is $g_A^{-1}(i)$. It follows that $\{W_i \longrightarrow V\}$ is a luin covering with factorizations $W_i \longrightarrow \mathcal{F}^{g^{-1}(i)} \longrightarrow \mathcal{F}$, as desired.

For the last claim, set $X = \coprod_i X_i$ and $h\colon X \longrightarrow I$ such that $h_{|X_i} \equiv i$. We have $X^{h^{-1}(i)} = X_i$. Using 4.14 we obtain

$$X^{\mathrm{P}} \cong (\coprod_{i\in I}(X^{h^{-1}(i)})^{\mathrm{P}})^{\mathrm{P}} \cong (\coprod_{i\in I} X_i^{\mathrm{P}})^{\mathrm{P}} \cong (\coprod_{i\in I}(\mathcal{F}^{g^{-1}(i)})^{\mathrm{P}})^{\mathrm{P}} \cong \mathcal{F}^{\mathrm{P}}$$

□

Remark 4.28. Let X be a locally Noetherian scheme and U_n be an increasing sequence of open subsets of X such that U_{n+1} contains the generic points of $X \setminus U_n$. Then $X = \bigcup_n U_n$. Indeed if $p \in X$ is a point, $\phi\colon \operatorname{Spec}(\mathcal{O}_{X,p}) \longrightarrow X$ the structure map and $C \subseteq X$ is a closed subset then the generic points of $\phi^{-1}(C)$ are the generic points of C contained in $\operatorname{Im}(\phi)$. In particular one can assume that X has finite dimension, in which case an induction on the dimension prove the claim.

Lemma 4.29. *Let X be a locally Noetherian scheme. Then there are finitely presented immersions $U_i \longrightarrow X$ with U_i affine and irreducible such that the map*

$$\phi\colon \coprod_i U_i \longrightarrow X$$

is surjective, quasi-compact and a monomorphism. In particular it is geometrically bijective and a P-isomorphism.

Proof. The last claim follows from 4.14. Given a locally Noetherian scheme X and a generic point ξ choose an open affine subset X_ξ of X which is irreducible and contains ξ (which will be its generic point). Notice that if ξ and η are two generic points of X then $X_\xi \cap X_\eta \neq \emptyset$ implies $\xi = \eta$. Set

$$V(X) = \coprod_{\xi \text{ generic point of } X} X_\xi \text{ and } Z(X) = X \setminus V(X)$$

So that $V(X)$ is open and $Z(X)$ is closed. The latter will be though of as a closed subscheme with reduced structure. Since X is locally Noetherian the map $Z(X) \longrightarrow X$ is a closed immersion of finite type. By induction set $V_{n+1}(X) = V(Z_n(X))$, $Z_{n+1}(X) = Z(Z_n(X))$, $V_0(X) = \emptyset$ and $Z_0(X) = X$. By construction all maps $Z_n(X) \longrightarrow X$ are closed immersion of finite type and $V_n(X) \longrightarrow X$ are immersion of finite type. Moreover

$$X \setminus Z_n(X) = \coprod_{k=0}^{n} V_k(X)$$

as sets. In conclusion the map

$$\coprod_n V_n(X) \longrightarrow X$$

is a monomorphism by construction and it is surjective by 4.28. It remains to show that it is quasi-compact. So let $U \subseteq X$ be a quasi-compact open subset. Since the union of $(X \setminus Z_n(X)) \cap U$ covers U, the previous sequence must stabilize. Moreover since $Z_n(X) \cap U$ is quasi-compact and Noetherian, it follows that $V_{n+1}(X) \cap U$ is a finite disjoint union of its irreducible components. This ends the proof. □

We are going to introduce some notation to explain next Lemma 4.30, which is a key ingredient in the proof of Theorem 1.1. A direct system of stacks $\{\mathcal{Z}_n\}_{n\in\mathbb{N}}$ is a chain of stacks $\mathcal{Z}_0 \longrightarrow \cdots \longrightarrow \mathcal{Z}_n \longrightarrow \mathcal{Z}_{n+1} \longrightarrow \cdots$ (see [10, Appendix A]). Such a sequence always admits a stack $\mathcal{Z}_\infty = \operatorname{colim}_n \mathcal{Z}_n$ as colimit (see [10, Proposition A.1 and Proposition A.5], [11, Remark A.3]). Moreover for an affine scheme U any object $U \longrightarrow \mathcal{Z}_\infty$ is induced by some $U \longrightarrow \mathcal{Z}_n$ (see [10, just before Proposition A.2]).

Assume that all $\mathcal{Z}_n$ are separated Deligne-Mumford stacks of finite type over a field k, so that they admit coarse moduli spaces $\mathcal{Z}_n \longrightarrow \overline{\mathcal{Z}_n}$. As a consequence of [10, Lemma 3.2] we have the following. The colimit $\mathcal{Z}_\infty \longrightarrow \operatorname{colim}_n(\overline{\mathcal{Z}_n}) = \overline{\mathcal{Z}_\infty}$ of the coarse moduli maps $\mathcal{Z}_n \longrightarrow \overline{\mathcal{Z}_n}$ is a coarse ind-algebraic space map in the sense of [10, Definition 3.1]. Moreover if the transition maps $\mathcal{Z}_n \longrightarrow \mathcal{Z}_{n+1}$ are finite and universally injective then so are the maps $\overline{\mathcal{Z}_n} \longrightarrow \overline{\mathcal{Z}_{n+1}}$: they are universally injective, thus quasi-finite, by [10, Lemma 3.2], they are proper because so are the coarse moduli maps $\mathcal{Z}_n \longrightarrow \overline{\mathcal{Z}_n}$. By [10, Proposition 2.9] finite and universally injective is the same as a composition of a finite universal homeomorphism and a closed immersion.

Lemma 4.30. *Let $\{\mathcal{Z}_n\}_{n\in\mathbb{N}}$ be a direct system of separated Deligne-Mumford stacks of finite type over k with finite and universally injective transition maps and colimit $\mathcal{Z}$. Then there are affine varieties $\{Y_i\}_{i\in\mathbb{N}}$ and a map*

$$\coprod_i Y_i \longrightarrow \overline{\mathcal{Z}}$$

where $\overline{(-)}$ denotes the corresponding ind-coarse moduli space, which is geometrically bijective and an epimorphism in the sur topology, so that $\coprod_i Y_i$ is a strong P-moduli space for $\overline{\mathcal{Z}}$. Moreover the functor of isomorphism classes of $\mathcal{Z}$, its Zariski, étale and fppf sheafifications all have the same strong P-moduli space of $\overline{\mathcal{Z}}$.

Proof. Set $\mathcal{U}_{n+1} = \mathcal{Z}_{n+1} \setminus \mathcal{Z}_n$. It is easy to see that $\overline{\mathcal{Z}_n} \amalg \overline{\mathcal{U}_{n+1}} \longrightarrow \overline{\mathcal{Z}_{n+1}}$ is geometrically bijective. Since $\overline{\mathcal{Z}}$ is the limit of the $\overline{\mathcal{Z}_n}$ the induced map $\coprod_n \overline{\mathcal{U}_n} \longrightarrow \overline{\mathcal{Z}}$ is geometrically bijective. Moreover it is an epimorphism in the sur topology because any map $U \longrightarrow \overline{\mathcal{Z}}$ from an affine scheme factors through some $\overline{\mathcal{Z}_n}$. Each $\overline{\mathcal{U}_n}$ is an algebraic space of finite type over k. By 4.14 and 4.18 the first part of the statement follows.

Now denote by $\mathcal{F}$ the functor of isomorphism classes of $\mathcal{Z}$ and by $\mathcal{F}^{sh}$ its sheafification for some of the topologies in the statement or $\mathcal{F}$ itself. The map $\mathcal{F}^{sh} \longrightarrow \overline{\mathcal{Z}}$ is geometrically bijective by definition of coarse ind-algebraic space (see [10, Definition 3.1]). It is also an epimorphism in the sur topology: a map $V \longrightarrow \overline{\mathcal{Z}}$ from a scheme factors Zariski locally through $\overline{\mathcal{Z}_n}$, sur locally through $\mathcal{Z}_n$ and therefore through $\mathcal{F}^{sh}$. From 4.14 we conclude that $(\mathcal{F}^{sh})^{\mathrm{P}} \cong \overline{\mathcal{Z}}^{\mathrm{P}}$. □

4.3. **P-schemes locally of finite type over a locally Noetherian scheme.**

We fix a locally Noetherian scheme S as base.

Remark 4.31. By 3.11 a geometrically bijective map $f\colon X \longrightarrow Y$ between schemes locally of finite type over S is quasi-compact and, therefore, of finite type.

Lemma 4.32. *Let X and Y be schemes locally of finite type over S and let $f\colon Y \longrightarrow X$ be a P-morphism over S. Then*

- *there exists a geometrically bijective morphism $Z \longrightarrow Y$ of finite type such that the composite P-morphism $Z \longrightarrow Y \longrightarrow X$ is induced by a scheme morphism $Z \longrightarrow X$;*
- *the map f is a P-isomorphism if and only if f is geometrically bijective.*

Proof. By 4.29 we can assume that S is affine and that X and Y are disjoint unions of affine schemes. In particular X and Y are separated over S. The first statement follows from 3.16 and 4.7, the second from 4.10. □

Corollary 4.33. *Two schemes X and Y locally of finite type over S are P-isomorphic if and only if there exist a scheme Z and geometrically bijective maps of finite type $Z \longrightarrow X$ and $Z \longrightarrow Y$.*

Proof. The "if part" follows directly from 4.32. Indeed Z is locally of finite type over S because, for instance, $Z \longrightarrow X$ is of finite type. Thus $Z \longrightarrow X$, $Z \longrightarrow Y$ are P-isomorphisms because they are geometrically bijective.

Let's focus on the "only if part". Let $f\colon Y \longrightarrow X$ be a P-isomorphism. By 4.32 there is a geometrically bijective morphism $Z \longrightarrow Y$ of finite type such that the composition $Z \longrightarrow Y \longrightarrow X$ is induced by a scheme morphism $g\colon Z \longrightarrow X$. In particular Z is locally of finite type over S. Moreover g is geometrically bijective and, by 4.31, of finite type. □

Lemma 4.34. *Let X be a locally Noetherian scheme and $C \subseteq X$ be a locally constructible subset. Then there exists a monomorphism $Z \longrightarrow X$ of finite type with image C. Moreover if $Z' \longrightarrow X$ is another map which is geometrically injective, locally of finite type and has image C then Z' and Z are P-isomorphic over X.*

Proof. The existence follows from 3.15 and 4.29. For the last statement notice that the projections $Z \times_X Z' \rightrightarrows Z, Z'$ are P-isomorphisms thanks to 4.32. □

Definition 4.35. In the situation of Lemma 4.34 we will say that a scheme is *P-isomorphic to C* if it is P-isomorphic to Z.

We conclude the section by an useful result for schemes over a field.

Lemma 4.36. *Let X and Y be schemes locally of finite type over k, $f\colon X \longrightarrow Y$ be a geometrically injective map and $x \in X$. Then, for every point $x \in X$,*

$$\dim \overline{\{x\}} = \dim \overline{\{f(x)\}} = \operatorname{degtr} k(x)/k$$

where degtr *denotes the transcendence degree. Moreover* $\dim X \leq \dim Y$ *and the equality holds if f is also surjective. In particular two schemes locally of finite type over k and P-isomorphic have the same dimension.*

Proof. The last two statements are a consequence of the first one and 4.33. The equality $\dim \overline{\{x\}} = \operatorname{degtr} k(x)/k$ is [9, Tag 02JX]. The equality $\dim \overline{\{x\}} = \dim \overline{\{f(x)\}}$ instead follows from the fact that, if $y = f(x)$, then $k(x)/k(y)$ is finite: the fiber map $X \times_Y k(y) \longrightarrow \operatorname{Spec} k(y)$ is non-empty, geometrically injective and locally of finite type, which easily implies that $X \times_Y k(y)$ is the spectrum of a local and finite $k(y)$-algebra. □

4.4. **Quotients by P-actions of finite groups.** In what follows G will denote a finite group.

Definition 4.37. Let X be an S-scheme. A *P-automorphism* of X is a P-morphism $f\colon X \longrightarrow X$ which is invertible, that is, there exists a P-morphism $f'\colon X \longrightarrow X$ with $f \circ f' = f' \circ f = \mathrm{id}_X$. A *P-action* of a finite group G on X means a group homomorphism $G \longrightarrow \mathrm{Aut}_S^P(X)$, where $\mathrm{Aut}_S^P(X)$ is the group of P-automorphisms of X. A P-morphism $g\colon X \longrightarrow Y$ between two S-schemes with a P-action of G is *equivariant* it is so in the category of P-schemes over S.

When we are given a P-action of a group G on a scheme X, a *geometric P-quotient* is a P-morphism $\pi\colon X \longrightarrow W$ of S-schemes such that:

(1) the map π is G-invariant, that is, for every $g \in G$, $\pi \circ g = \pi$,
(2) the map π is universal among G-invariant P-morphisms, that is, if $\pi'\colon X \longrightarrow W'$ is another G-invariant P-morphism of S-schemes, then there exists a unique P-morphism $h\colon W \longrightarrow W'$ such that $h \circ \pi = \pi'$,
(3) for each algebraically closed field K over S, the map $X(K)/G \longrightarrow W(K)$ is bijective.

A P-morphism $\pi\colon X \longrightarrow W$ of S-schemes is a *strong P-quotient* if it is G-invariant and the induced map $X^P/G \longrightarrow W^P$ is a strong P-moduli space, where X^P/G is the functor $U \longmapsto X^P(U)/G$.

Remark 4.38. Recall that for S-schemes X and Y one has

$$\mathrm{Hom}_S^{\mathrm{P}}(X,Y) \cong \mathrm{Hom}_S(X^{\mathrm{P}}, Y^{\mathrm{P}})$$

by 4.6, 5), more precisely $(-)^{\mathrm{P}}\colon P\text{-}\mathbf{Sch}/S \longrightarrow \mathrm{Sh}_{\mathrm{sur}}(\mathbf{Sch}'/S)$ is fully faithful by 4.12. In particular: a P-action of G on X is just an action of G on X^{P}; a G-invariant map $X \longrightarrow W$ is a G-invariant map $X^{\mathrm{P}} \longrightarrow W^{\mathrm{P}}$, that is a map $X^{\mathrm{P}}/G \longrightarrow W^{\mathrm{P}}$; an equivariant P-morphism $Y \longrightarrow X$ is an equivariant morphism $X^{\mathrm{P}} \longrightarrow Y^{\mathrm{P}}$. Moreover it follow easily that $X \longrightarrow W$ is a geometric P-quotient (resp. strong P-quotient) if and only if $X^{\mathrm{P}}/G \longrightarrow W^{\mathrm{P}}$ is a P-moduli space (resp. a strong P-moduli space).

Proposition 4.39. *Let X be an S-scheme with a P-action of G and $X \longrightarrow W$ be a G-invariant P-morphism over S such that $X(K)/G \longrightarrow W(K)$ are bijective for all algebraically closed fields K over S. If $X \longrightarrow W$ is an epimorphism in the sur topology then $X \longrightarrow W$ is a strong P-quotient. This is the case, for example, if $X \longrightarrow W$ is a locally of finite presentation map of S-schemes.*

Proof. Set $F = X^{\mathrm{P}}/G$. By hypothesis the map $F \longrightarrow W^{\mathrm{P}}$ is geometrically bijective, that is, by 4.14, the map $F^{\mathrm{P}} \longrightarrow W^{\mathrm{P}}$ is a monomorphism. The previous map is an isomorphism, that is $X \longrightarrow W$ is a strong P-quotient, if and only if $F \longrightarrow W^{\mathrm{P}}$ is an epimorphism in the sur topology. This is true if $X \longrightarrow W$ is an epimorphism as well. Notice that $X \longrightarrow W$ is surjective because the map $X(K) \longrightarrow X(K)/G \longrightarrow W(K)$ is so for all algebraically closed fields K over S. Therefore if $X \longrightarrow W$ is locally of finite presentation then this map is a sur covering. □

Corollary 4.40. *Let X be an S-scheme with an (usual) action of G. If $X \longrightarrow W$ is a G-invariant map of S-schemes, it is locally of finite presentation and $X(K)/G \longrightarrow W(K)$ is an isomorphism for all algebraically closed fields K, then it is also a strong P-quotient. In particular geometric quotients are strong P-quotients.*

Lemma 4.41. *Consider a diagram*

$$\begin{array}{ccc} & & Z \\ & \overset{\tilde{f}}{\nearrow} & \downarrow{\scriptstyle u} \\ X & \xrightarrow{f} & Y \end{array}$$

of S-schemes where f is a P-morphism and u is a locally finitely presented and geometrically injective map. If $f(X) \subseteq u(Z)$ as sets then there exists a unique dashed P-morphism $\tilde{f}$ making the above diagram commutative in P-$\mathbf{Sch}/S$.

Proof. Since u is locally of finite type we have that $f\colon X_F \longrightarrow Y_F$ has value in $Z_F \subseteq Y_F$. We just have to show that $X_F \longrightarrow Z_F$ is a P-morphism. In particular we can assume that f is induced by a map of schemes. In this case we obtain a map $X \times_Y Z \longrightarrow X$ which is locally of finite presentation and, by hypothesis, surjective. Thus it is a sur covering of X and the map $X \times_Y Z \longrightarrow Z$ lifts $(X \times_Y Z)_F \longrightarrow X_F \longrightarrow Z_F$. □

Lemma 4.42. *Let X be a scheme of finite type over a field k endowed with a P-action of a finite group G and let $U \subset X$ be an open subset with $\dim(X \setminus U) < \dim X$. Then there exists an open subset $V \subset U$ with $\dim(X \setminus V) < \dim X$ which is G invariant, a finite universal homeomorphism $h\colon V' \longrightarrow V$ and an action of G on V' making h equivariant.*

Proof. Notice that the condition $\dim(X \setminus U) < \dim X$ just means that U meets the irreducible components of X of maximal dimension $\dim X$. From 4.32, there exists a geometrically bijective map $Z_g \longrightarrow U$ of finite type such that $Z_g \longrightarrow U \xrightarrow{g} X$ is induced by a scheme morphism. Taking the fiber products of the Z_g over U we can find a common map $h\colon Z \longrightarrow U$. Call $h_g\colon Z \longrightarrow X$ the lifting of $U \xrightarrow{g} X$, with $h_{\mathrm{id}} = h$.

We first show that G permutes the generic points of the irreducible components of X of dimension $d = \dim X$. If ξ is a generic point of such a component, then $\xi \in U$, $g(\xi) = h_g(h^{-1}(\xi))$ and using 4.36, it follows that $d = \dim \overline{\{\xi\}} = \dim \overline{\{g(\xi)\}}$. Since $\dim X = d$ we can also conclude that $g(\xi)$ is a generic point.

The maps $h_g\colon Z \longrightarrow X$ are quasi-compact, quasi-separated and geometrically injective. By [9, Tag 02NW] there exists an open dense subset W of U such that $h_g^{-1}(W) \longrightarrow W$ is finite for all g. Set $W' = h^{-1}(W)$. In particular $h\colon W' \longrightarrow W$ is a finite universal homeomorphism. Notice that $h(h_g^{-1}(W) \cap W') = W \cap g^{-1}(W)$ as sets and it is an open subset of W. Consider $V := \bigcap_{g \in G} g(W)$, which is open in W and set $V' := h^{-1}(V) \longrightarrow V$, which is a finite universal homeomorphism. Notice that V contains the generic points of the irreducible components of maximal dimension. Therefore $\dim(X \setminus V) < \dim X$. Moreover the composition $V' \subseteq W' \xrightarrow{h_g} X$, which set-theoretically is $V' \longrightarrow V \xrightarrow{g} X$, factors through V and $h_g\colon V' \longrightarrow V$ is surjective. Since this map is a restriction of the finite and geometrically injective map $h_g^{-1}(W) \longrightarrow W$, we can conclude that $h_g\colon V' \longrightarrow V$ is a finite universal homeomorphism.

We now modify V' in order to define an action on it. Notice that if $\widetilde{V}$ is an open subset of V with $\dim(X \setminus \widetilde{V}) < \dim X$, by discussion above it always contains a G-invariant open with the same property and we can always replace V by it. Moreover we can always assume $X = V$. In conclusion we can shrink as much as we want around the generic points of X of maximal dimension. In particular we can assume that $V = X$ and V' are a disjoint union of affine integral varieties of the same dimension.

Let $G(X)$ the generic points of X and for $\xi \in G(X)$ let η_ξ the generic point of V' mapping to ξ. For all $\xi \in G(X)$ set also K_ξ for the perfect closure of $k(\xi)$. Recall that if $L/k(\xi)$ is a purely inseparable extension then there exists a unique $k(\xi)$ linear map $L \longrightarrow K_\xi$. Since $V' \longrightarrow X$ is a finite universal homeomorphism it follows that $k(\xi) \longrightarrow k(\eta_\xi)$ is finite and purely inseparable. So we can assume $k(\eta_\xi) \subseteq K_\xi$. We have that G permutes $G(X)$ and, since $h_g \colon V' \longrightarrow X$ is a finite universal homeomorphism, it also induces a finite purely inseparable extension $k(g(\xi)) \longrightarrow k(\eta_\xi)$. In particular there exists a unique map $\phi_{g,\xi}$ making the following diagram commutative:

$$\begin{array}{ccccc} & & k(\eta_\xi) & \longrightarrow & K_\xi \\ & \overset{h_g}{\nearrow} & & & \downarrow{\scriptstyle \phi_{g,\xi}} \\ k(g(\xi)) & \longrightarrow & k(\eta_{g(\xi)}) & \longrightarrow & K_{g(\xi)} \end{array}$$

We claim that the two maps $\phi_{ab,\xi}, \phi_{a,b(\xi)} \circ \phi_{b,\xi} \colon K_\xi \longrightarrow K_{ab(\xi)}$ are the same map. Let $\alpha, \beta \colon \operatorname{Spec} K_{ab(\xi)} \longrightarrow \operatorname{Spec} K_\xi$ be the corresponding maps. By hypothesis they coincide as P-morphisms if composed with $\operatorname{Spec} K_\xi \longrightarrow X$. If $\overline{K}$ is an algebraic closure of $K_{ab(\xi)}$ then the two maps

$$\operatorname{Spec} \overline{K} \longrightarrow \operatorname{Spec} K_{ab(\xi)} \rightrightarrows \operatorname{Spec} K_\xi \longrightarrow \operatorname{Spec} k(\xi)$$

coincide. Using the usual properties of purely inseparable extensions and the perfect closure we can conclude that $\alpha = \beta$. In particular all maps $\phi_{g,\xi}$ are isomorphisms. If we set $\tilde{K}_\xi$ as the composite of all extensions $\phi_{g,\xi}^{-1}(k(\eta_{g(\xi)}))$ it follows that $\tilde{K}_\xi / k(\eta_\xi)$ is finite and purely inseparable and $\phi_{g,\xi}$ restricts to an isomorphism $\tilde{K}_\xi \longrightarrow \tilde{K}_{g(\xi)}$. If V'_ξ is the irreducible component of η_ξ we can find an open dense U_ξ and a finite universal homeomorphism $U' \longrightarrow U_\xi$ with U' integral and fraction field $\tilde{K}_\xi$. Shrinking X we can assume $k(\eta_\xi) = \tilde{K}_\xi$. The map $\phi_{g^{-1},g(\xi)}$ yield a generic map $\psi_{g,\xi} \colon V'_\xi \longrightarrow V'_{g(\xi)}$ and shrinking again X we can assume it is defined everywhere and, more generally, that it defines an action of G on V'.

The maps $V' \xrightarrow{\psi_g} V' \xrightarrow{h} X$ and $V' \xrightarrow{h_g} X$ coincide in the generic points and therefore they are generically the same because V' is reduced. Again shrinking X we can assume they coincide. But this exactly means that the P-action of G on V' obtained conjugating the P-isomorphism $V' \longrightarrow X$ is induced by the maps ψ_g on V'. By 4.6, (7) we can conclude that the collection of maps $\{\psi_g\}_g$ defines a "genuine" action of G on V', which ends the proof. □

Proposition 4.43. *Let X be a scheme locally of finite type over a field k and with a P-action of a finite group G. Then there exist a locally of finite type scheme Y with an action of G, a geometrically bijective map $Y \longrightarrow X$ of finite type which is G-equivariant and a decomposition of $Y = \coprod_i Y_i$ into G-invariant open affine subsets.*

Proof. From 4.29 and 4.41 we can assume $X = \coprod X_q$ where the X_q are integral schemes of finite type over k. From 4.32, there exists a geometrically bijective map $\phi \colon Z_g \longrightarrow X$ of finite type such that $Z_g \longrightarrow X \xrightarrow{g} X$ is induced by a scheme morphism $h_g \colon Z_g \longrightarrow X$. Taking the fiber products of the Z_g over X we can find a common map $\phi\colon Z \longrightarrow X$. Since $g(X_q) = h_g(\phi^{-1}(X_q))$, this is a locally constructible set of X. Moreover since X_q is quasi-compact and ϕ is of finite type, $g(X_q)$ is contained in a quasi-compact open of X. In particular $Z_q = \bigcup_g g(X_q)$ is a locally constructible subset of X contained in a quasi-compact open subset. Moreover it is G-invariant. We use the notation in 3.14 with I the index set of the Z_q. Let $q \in I$ and consider indexes $Z_q \subseteq X_{q_1} \amalg \cdots \amalg X_{q_l}$. We claim that $Z_q \cap Z_{q'} \neq \emptyset$ implies that $q' = q_i$ for some i. From this and 3.14 it will follow that, for $J \subseteq I$ finite, Z_J is locally constructible and

$$X = \coprod_{J \subseteq I \text{ finite}} Z_J$$

as sets. If $Z_q \cap Z_{q'} \neq \emptyset$ there exist $g, h \in G$ such that $g(X_q) \cap h(X_{q'}) \neq \emptyset$, that is $\emptyset \neq h^{-1}g(X_q) \cap X_{q'} \subseteq Z_q \cap X_{q'}$, from which the claim follows.

For all J finite, since Z_J is locally constructible, we have a monomorphism $Y_J \longrightarrow X$ of finite type onto Z_J by 3.15. Since Z_J is contained in a quasi-compact open of X it follows that Y_J is quasi-compact, that is of finite type. By construction the Z_J are G-invariant and, by 4.41, we can lift the P-action of G on X to a P-action of G on Y_J.

The argument above shows that we can replace X by a scheme of finite type. We can also assume X reduced and, by 4.29, also separated. Consider the open V and the map $h\colon V' \longrightarrow V$ obtained from 4.42. By a dimension argument and an induction on $\dim X$ we can assume $V = X$ and that G has a genuine action on X inducing the P-action. Consider a dense affine open subset W of X and replacing it by $\bigcap_g g(W)$ so that it is also G-invariant. Again since $\dim(X \setminus W) < \dim X$ we can assume $X = W$ and we are done. □

Theorem 4.44. *Let X be a scheme (locally) of finite type over a field k endowed with a P-action of a finite group G. Then X has a strong P-quotient $X \longrightarrow Y$ with Y (locally) of finite type over k. Moreover if X is P-isomorphic to a countable disjoint union of affine k-varieties then so is the strong P-quotient Y.*

Proof. Notice that if $U = \coprod_{n \in \mathbb{N}} U_n$ is P-isomorphic to $V = \coprod_{i \in I} V_i$ where V_i and U_n are schemes of finite type over k with $V_i \neq \emptyset$ then I is at most countable. Indeed there exist geometrically bijective maps of finite type $\phi\colon Z \longrightarrow U$ and $\psi\colon Z \longrightarrow V$ thanks to 4.33. Thus one can assume $Z = U = V$ and notice that the sets $\{i \in I \mid U_n \cap V_i \neq \emptyset\}$ are finite and cover I. Thanks to the previous observation and by 4.43 we can assume $X = \operatorname{Spec} A$ affine and that the P-action of G on X is actually an action. Then $X \longrightarrow X/G = \operatorname{Spec}(A^G)$ is a geometric quotient and A^G is of finite type over k. By 4.40 the map $X \longrightarrow X/G$ is a strong P-quotient. □

5. Motivic integration on schemes locally of finite type

In this section we construct a modified Grothendieck ring using the theory of P-schemes.

Definition 5.1. The *modified Grothendieck ring of varieties*, $K_0^{\mathrm{mod}}(\mathbf{Var}/k)$, is the free abelian group generated by the P-isomorphism classes of k-varieties modulo the relation $[X] = [U] + [V]$ if X and $U \amalg V$ are P-isomorphic. The product structure is given by $[X][Y] := [X \times Y]$.

In particular, the usual scissor relation holds: if $Y \subset X$ is a closed subvariety, then $[X] = [Y] + [X \setminus Y]$. Moreover, if X and Y are P-isomorphic, then $[X] = [Y]$.

Definition 5.2. We denote by $\mathbb{L}$ the class of an affine line $[\mathbb{A}^1_k]$ in $K_0^{\mathrm{mod}}(\mathbf{Var}/k)$. We define $\mathcal{M}_k^{\mathrm{mod}}$ to be the localization of $K_0^{\mathrm{mod}}(\mathbf{Var})$ by $\mathbb{L}$. For a positive integer l, we define $\mathcal{M}_k^{\mathrm{mod},l}$ to be $\mathcal{M}_k^{\mathrm{mod}}[\mathbb{L}^{1/l}] = \mathcal{M}_k^{\mathrm{mod}}[x]/(x^l - \mathbb{L})$. In this ring, we have fractional powers $\mathbb{L}^r$, $r \in \frac{1}{l}\mathbb{Z}$ of $\mathbb{L}$. We then define a completion $\hat{\mathcal{M}}_k^{\mathrm{mod},l}$ of $\mathcal{M}_k^{\mathrm{mod},l}$as follows. Let $F_m \subset \mathcal{M}_k^{\mathrm{mod},l}$ be the subgroup generated by the elements $[X]\mathbb{L}^r$ with $\dim X + r \leq -m$. We define

$$\hat{\mathcal{M}}_k^{\mathrm{mod},l} := \varprojlim_m \mathcal{M}_k^{\mathrm{mod},l}/F_m,$$

which inherits the ring structure since $F_m F_n \subset F_{m+n}$. When $l = 1$, we abbreviate $\mathcal{M}_k^{\mathrm{mod},l}$ and $\hat{\mathcal{M}}_k^{\mathrm{mod},l}$ to $\mathcal{M}_k^{\mathrm{mod}}$ and $\hat{\mathcal{M}}_k^{\mathrm{mod}}$ respectively.

Recall that a P-morphism $X \longrightarrow Y$ of schemes induces a map $|X| \longrightarrow |Y|$ on the set of points.

Definition 5.3. Let X be a scheme locally of finite type over k, $l \in \mathbb{Z} \setminus \{0\}$ and $f\colon X \longrightarrow \frac{1}{l}\mathbb{Z}$ be a function, that is a map of sets from the set of points $|X|$ of X to $\frac{1}{l}\mathbb{Z}$. The map f is called *integrable* if there are non-empty schemes $\{X_i\}_{i\in I}$ of finite type over k and a P-isomorphism $\phi\colon \coprod_i X_i \longrightarrow X$ such that $f \circ \phi$ is constant on all X_i and, for all $n \in \mathbb{Z}$, there are at most finitely many $i \in I$ such that $\dim X_i + f(\phi(X_i)) > n$.

We define the *integral* $\int_X \mathbb{L}^f \in \hat{\mathcal{M}}_k^{\mathrm{mod},l} \cup \{\infty\}$ of a function $f\colon X \longrightarrow \frac{1}{l}\mathbb{Z}$ as follows. If f is integrable,

$$\int_X \mathbb{L}^f := \sum_{i\in I} [X_i]\mathbb{L}^{f(\phi(X_i))} \in \hat{\mathcal{M}}_k^{\mathrm{mod},l}.$$

Otherwise$\int_X \mathbb{L}^f := \infty$.

Notice that, if we follow the usual convention that $\dim \emptyset = -\infty$, in the definition of integrability and of integrals we don't have to assume that the schemes X_i are non empty.

The following lemma shows that the notion of integrability and the integral itself do not depend on the choice of the k-schemes X_i.

Lemma 5.4. *Let X be a scheme locally of finite type over k, $l \in \mathbb{Z} \setminus \{0\}$ and $f\colon X \longrightarrow \frac{1}{l}\mathbb{Z}$ be a function. Let $\{Y_j\}_{j\in J}$ be non-empty schemes of finite type over k and $\phi\colon Y = \coprod_j Y_j \longrightarrow X$ be a P-isomorphism such that $f \circ \phi$ is constant on all Y_j. If f is integrable, then for each $n \in \frac{1}{l}\mathbb{Z}$, there are at most finitely many $j \in J$ such that* $\dim Y_j + f(Y_j) > n$ *and*

$$\int_X \mathbb{L}^f = \sum_{j\in J} [Y_j]\mathbb{L}^{f(\phi(Y_j))} \in \hat{\mathcal{M}}_k^{\mathrm{mod},l}.$$

Proof. Following the notation of Definition 5.3 we can assume $X = \coprod_i X_i$. By 4.33 there exist a scheme Z and geometrically bijective maps of finite type $\alpha\colon Z \longrightarrow \coprod_i X_i$, $\beta\colon Z \longrightarrow \coprod_j Y_j$. In particular $\alpha^{-1}(X_i)$ and $\beta^{-1}(Y_j)$ are of finite type and those maps preserve dimension thanks to 4.36. We can therefore assume $Z = X = Y$. Set

$$I_n = \{i \in I \mid f(X_i) + \dim X_i > n\} \text{ and } J_n = \{j \in J \mid f(Y_j) + \dim Y_j > n\}.$$

Given $j \in J$ take a generic point $\eta_j \in Y_j$ with $\dim \overline{\{\eta_j\}} = \dim Y_j$ and let $s_j \in I$ be such that X_{s_j} contains the point η_j. We have $f(X_{s_j}) = f(Y_j)$ and, by 4.36,

$\dim Y_j \leq \dim X_{s_j}$. In particular $s\colon J \longrightarrow I$ maps J_n into I_n and, in order to show that J_n is finite, it is enough to show that s has finite fibers. The result follows from

$$X_i = \coprod_{j \in J} X_i \cap Y_j$$

and the fact that $s_j = i$ implies that $X_i \cap Y_j \neq \emptyset$.

For the last equality, it is enough to use the (finite) sums

$$[X_i] = \sum_j [X_i \cap Y_j] \text{ and } [Y_j] = \sum_i [X_i \cap Y_j]$$

in $K_0^{\mathrm{mod}}(\mathbf{Var}/k)$ and that, if $X_i \cap Y_j \neq \emptyset$ then $f(X_i) = f(X_i \cap Y_j) = f(Y_j)$. □

Definition 5.5. Let $\mathcal{F}\colon \mathbf{Sch}'/k \longrightarrow \mathbf{Set}$ be a functor with a scheme locally of finite type X as strong P-moduli space and $f\colon \mathcal{F} \longrightarrow \frac{1}{l}\mathbb{Z}$ be a function, which is induced by $f_X\colon X \longrightarrow \frac{1}{l}\mathbb{Z}$ (see 4.25). The map f is called *integrable* if f_X is so. Moreover we set $\int_{\mathcal{F}} \mathbb{L}^f = \int_X \mathbb{L}^{f_X}$.

If Y is a scheme locally of finite type over k and $C \subseteq Y$ a locally constructible subset a function $f\colon C \longrightarrow \frac{1}{l}\mathbb{Z}$ is just a function of sets $|C| \longrightarrow \frac{1}{l}\mathbb{Z}$. We define *constructibility* and *integrability* for $f\colon C \longrightarrow \frac{1}{l}\mathbb{Z}$ as the ones for $f\colon X \longrightarrow \frac{1}{l}\mathbb{Z}$, where X is a scheme P-isomorphic to C. Moreover we set $\int_C \mathbb{L}^f = \int_X \mathbb{L}^f$.

Notice that, by 4.21, in the above definition the second definition is a particular case of the previous one.

Proposition 5.6. *Let $f\colon X \longrightarrow \frac{1}{l}\mathbb{Z}$ be a function from a scheme locally of finite type over k. Then f is integrable if and only if the following three conditions are satisfied: (1) f is bounded above, (2) for all $n \in \frac{1}{l}\mathbb{Z}$ the set $f^{-1}(n)$ is locally constructible and P-isomorphic to a scheme of finite type over k and (3)*

$$n - \dim(f^{-1}(-n)) \longrightarrow +\infty \textit{ for } \frac{1}{l}\mathbb{Z} \ni n \longrightarrow +\infty$$

where we use the usual convention $\dim \emptyset = -\infty$.

Proof. In both cases we can assume $X = \coprod_i X_i$ with f constant on all X_i and X_i of finite type and non-empty. If f is integrable then

$$\{i \in I \mid f(X_i) = n\} \subseteq \{i \in I \mid f(X_i) + \dim X_i > n - 1\}$$

is finite, that is $f^{-1}(n)$ is P-isomorphic to a scheme of finite type. We can therefore assume $X = \coprod_{n \in \frac{1}{l}\mathbb{Z}} X_n$ with $X_n = f^{-1}(n)$ (and allowing $X_n = \emptyset$). By 5.4 integrability means that the sets $I_m = \{n \in \frac{1}{l}\mathbb{Z} \mid n - \dim X_{-n} < m\}$ are finite. The limit in the statement means that all I_m are bounded above. Finally if f is bounded above then all I_m are bounded below. Conversely if I_0 is bounded below then f is bounded above. □

Remark 5.7. If we are given a continuous ring homomorphism $\hat{\mathcal{M}}_k^{\mathrm{mod},l} \longrightarrow R$ of complete topological rings and continue to denote the image of $\mathbb{L}$ in R by $\mathbb{L}$, then we can similarly define integrals in $R \cup \{\infty\}$. Of course, these integrals coincide with the images of the corresponding integrals defined in $\hat{\mathcal{M}}_k^{\mathrm{mod},l} \cup \{\infty\}$.

6. Some results on power series rings

We collect in this section various results and notations about power series rings.

Lemma 6.1. *Let R be a ring and S be an R-algebra. Let M be an $R[[t]]$-module and M_t its localization by t, which is an $R((t)) = R[[t]]_t$-module. Then we have*

$$(M \otimes_{R[[t]]} S[[t]])_t \cong M_t \otimes_{R((t))} S((t)).$$

Proof. This follows from

$$(M \otimes_{R[[t]]} S[[t]]) \otimes_{S[[t]]} S[[t]]_t \cong M_t \otimes_{R[[t]]_t} S[[t]]_t.$$

□

Definition 6.2. Let R be a ring and S be an R-algebra. For an $R[[t]]$-module M, we define the *complete tensor product* as

$$M \hat{\otimes}_R S = M \otimes_{R[[t]]} S[[t]].$$

Remark 6.3. If N is an $R((t))$-module then by 6.1 we have

$$N \hat{\otimes}_R S \cong N \otimes_{R((t))} S((t)).$$

In particular if M is an $R[[t]]$-module and S an R-algebra then we have identifications

$$(M \hat{\otimes}_R S)_t \cong M_t \otimes_{R((t))} S((t)) \cong M_t \hat{\otimes}_R S.$$

Lemma 6.4. *Let R be a ring, S be a Noetherian R-algebra and M be a finitely generated $R[[t]]$-module. Then $M \otimes_R S \longrightarrow M \hat{\otimes}_R S$ is the completion with respect to the ideal $(t) \subseteq R[[t]]$, that is we have a natural isomorphism*

$$\varprojlim_{n \in \mathbb{N}} ((M/t^n M) \otimes_R S) \cong M \otimes_{R[[t]]} S[[t]].$$

Proof. The ring $S[[t]]$ is Noetherian and t-adically complete. Since $M \otimes_{R[[t]]} S[[t]]$ is a finitely generated $S[[t]]$-module, it is t-adically complete and the projective limit of

$$N_n := \big(M \otimes_{R[[t]]} S[[t]]\big) \otimes_{S[[t]]} (S[[t]]/(t^n)) \quad (n \in \mathbb{N}).$$

Since

$$S[[t]]/(t^n) \cong S[[t]] \otimes_{R[[t]]} (R[[t]]/(t^n)) \cong R[[t]]/(t^n) \otimes_R S,$$

we have

$$\begin{aligned} N_n &\cong \big(M \otimes_{R[[t]]} S[[t]]\big) \otimes_{R[[t]]} (R[[t]]/(t^n)) \\ &\cong (M/t^n M) \otimes_{R[[t]]/(t^n)} (S[[t]]/(t^n)) \\ &\cong (M/t^n M) \otimes_R S. \end{aligned}$$

The lemma follows. □

Remark 6.5. By [10, Lemma 2.4] if S is a finite and finitely presented R-algebra then

$$\omega_{S/R} \colon R[[t]] \otimes_R S \longrightarrow S[[t]]$$

is an isomorphism. In particular $M \hat{\otimes}_R S \cong M \otimes_R S$ for all $R[[t]]$-modules M.

Lemma 6.6. *Let R be a ring, $k > 0$ and $g \in R[[s]]^*$. Then there exists a unique map $R[[t]] \longrightarrow R[[s]]$ of R-algebras mapping t to $s^k g$ and $1, s, \ldots, s^{k-1}$ is an $R[[t]]$-basis of $R[[s]]$. In particular if $R \longrightarrow R'$ is a map of rings then $R[[s]] \hat{\otimes}_R R' (= R[[s]] \otimes_{R[[t]]} R'[[t]]) \cong R'[[s]]$ and $R((s)) \hat{\otimes}_R R' \cong R'((s))$.*

Proof. There are compatible maps $R[t]/(t^n) \longrightarrow R[s]/(s^n)$ mapping t to $s^k g$ and passing to the limit we get a map $R[[t]] \longrightarrow R[[s]]$. Uniqueness is easy to prove. Consider the map

$$\phi\colon R[[t]]^k \longrightarrow R[[s]]$$

mapping the canonical basis to $1, s, \dots, s^{k-1}$. Notice that $R[[s]]/t^n R[[s]] \cong R[s]/(s^{nk})$ because g is invertible. Thus tensoring the above map by $R[[t]]/(t^n)$ we obtain a map

$$\phi_n\colon (R[t]/(t^n))^k \longrightarrow R[s]/(s^{nk}).$$

In order to show that ϕ is an isomorphism it is enough to show that all ϕ_n are isomorphisms. Since ϕ_n is a map between free R-modules of the same rank, it is enough to show that ϕ_n is surjective. By Nakayama's lemma we can assume $n = 1$, where the result is clear.

Using that ϕ is an isomorphism it is easy to conclude that the map $R[[s]]\hat{\otimes}_R R' \longrightarrow R'[[s]]$ is an isomorphism. Since $t = s^k g$ we also have $R[[s]]_t = R((s))$, so that also the last isomorphism holds. □

Lemma 6.7. *Let R be a ring, $k > 0$, $\zeta_1, \zeta_2 \in R[[s]]^*$ and consider $R[[s_i]]$ as an $R[[t]]$ module via $R[[t]] \longrightarrow R[[s]]$, $t \longmapsto s^k \zeta_i$ for $i = 1, 2$. If $\sigma\colon R((s_1)) \longrightarrow R((s_2))$ is an isomorphism of $R((t))$-algebras then, up to modding out R by finitely many nilpotents, we have that $\sigma(R[[s_1]]) = R[[s_2]]$, more precisely there exists $u \in R[[s_2]]^*$ such that $\sigma(s_1) = us_2$. Moreover $\sigma_{|R[[s_1]]} : R[[s_1]] \longrightarrow R[[s_2]]$ is the unique R-linear map sending s_1 to us_2.*

Proof. From 6.6 we see that $R[[s_i]]$ is free of rank k over $R[[t]]$, in particular $R[[t]] \subseteq R[[s_i]]$ is an integral extension. Notice moreover that $R[[s_i]]_t = R((s_i))$. Set $\sigma(s_1) = \sum_{m\in\mathbb{Z}} \sigma_m s_2^m \in R((s_2))$. If R is a field then $\sigma(s_1) \in s_2 R[[s_2]]^*$: $R[[s_i]]$ is a DVR with maximal ideal (s_i) and it is the integral closure of $R[[t]]$ inside $R((s_i))$, so that

$$\sigma_{|R[[s_1]]}\colon R[[s_1]] \xrightarrow{\cong} R[[s_2]] \text{ and } (\sigma(s_1)) = (s_2)$$

This means that all the σ_m for $m \leq 0$ lie in all the prime ideals, that is they are nilpotent, and no prime ideal contains σ_1, that is σ_1 is invertible. Modding out finitely many nonzero σ_m with $m \leq 0$ (there are at most finitely many of them), we can assume that $\sigma(s_1) = us_2$ as in the statement. Since $R[[s_1]]$ is generated by s_1 as an $R[[t]]$ algebra it also follows that $\sigma(R[[s_1]]) \subseteq R[[s_2]]$. Doing the same for σ^{-1} one also gets the equality. The last statement follows from 6.6. □

Lemma 6.8. *Let R be a ring, $k > 0$ and $\zeta \in R[[s]]^*$. Consider $R[[s]]$ as an $R[[t]]$ module via $R[[t]] \longrightarrow R[[s]]$, $t \longmapsto s^k\zeta$ and assume that $R((s))$ has a structure of G-torsor over $R((t))$, where G is a finite group. Then $k = |G|$ and, up to modding out R by finitely many nilpotents, we have that:*

1. *for all $g \in G$ we have that $g(R[[s]]) = R[[s]]$, more precisely there exists $u_g \in R[[s]]^*$ such that $g(s) = u_g s$ and $g_{|R[[s]]} : R[[s]] \longrightarrow R[[s]]$ is the unique R-linear map sending s to $u_g s$;*
2. *if $H < G$ is a subgroup and $s' = \prod_{h\in H} h(s)$ then $s' = s^{|H|}v$ with $v \in R[[s]]^*$, the map $R[[y]] \longrightarrow R[[s]]$, $y \longmapsto s'$ is an isomorphism onto $R[[s]]^H$, $(R[[s]]^H)_t = (R[[s]]_t)^H$ and $t = s'^{|G|/|H|}w$ with $w \in R[[s']]^*$. In particular $R[[s]]$ is a free $R[[s]]^H$-module of rank $|H|$ and $R[[s]]^H$ is a free $R[[t]]$-module of rank $|G|/|H|$.*

Proof. From 6.6 we see that $R[[s]]$ is free of rank k over $R[[t]]$. Since $R[[s]]_t/R((t))$ is a G-torsor we can conclude that $k = |G|$. Point (1) follows from 6.7. Let us consider point (2). We have $s' = s^{|H|}v$ where $v = \prod_h u_h \in R[[s]]^*$ for $u_h \in R[[s]]^*$ as in (1). By 6.6 the map $\phi\colon R[[y]] \longrightarrow R[[s]]$, $\phi(y) = s'$, is well defined, injective and $1, s, \dots, s^{|H|-1}$ is an $R[[y]]$-basis. Since $s' \in R[[s]]^H$ it is easy to see that ϕ maps into $R[[s]]^H$. Since $R[[s]]_{s'} = R((s))$ we have

$$R((y)) \subseteq R((s))^H = (R[[s]]^H)_{s'} \subseteq R((s)).$$

Since $R((s))/R((s))^H$ is an H-torsor the $R((s))^H$-module $R((s))$ is projective of rank $|H|$ and generated by $1, s, \dots, s^{|H|-1}$, which is therefore a $R((s))^H$-basis: the induced map

$$(R((s))^H)^{|H|} \longrightarrow R((s))$$

is a surjective map of projective $R((s))^H$-modules of the same rank, thus an isomorphism. Let $x \in R[[s]]^H \subseteq R[[s]]$ and write

$$x = x_0 + x_1 s + \cdots + x_{|H|-1}s^{|H|-1} \text{ with } x_i \in R[[y]].$$

Since we also have $x_i \in R((s))^H$ and the writing is unique we conclude that $x_1 = \cdots = x_{|H|-1} = 0$ and $x = x_0$ in $R((s))^H$. The injectivity of $R[[y]] \longrightarrow R((s))^H$ implies that $x \in R[[y]]$. This shows that $R[[s']] = R[[s]]^H$ and

$$(R[[s]]^H)_t = R((s')) = R((s))^H = (R[[s]]_t)^H.$$

Finally since $t \in R[[s]]^H$ we have $t = s'^b q$, where $q \in R[[s']]$, $q(0) \neq 0$. Thus $t = s^{b|H|}v^b q = s^{|G|}\zeta$. Looking at the first non vanishing coefficient we conclude that $b|H| = |G|$ and that q is invertible. □

Lemma 6.9. *If $\{U_i\}_i$ is a Zariski covering of $\operatorname{Spec} R((t))$ for some ring R then there exists a ubi covering $\{\operatorname{Spec} R_j \longrightarrow \operatorname{Spec} R\}_j$ such that $\operatorname{Spec} R_j((t)) \longrightarrow \operatorname{Spec} R((t))$ factors through some of the U_i.*

Proof. We can assume $U_i = \operatorname{Spec} R((t))_{s_i}$ for $s_1, \dots, s_n \in R[[t]]$ such that $(s_1, \dots, s_n) = R((t))$. This means there exist $a_1, \dots, a_n \in R[[t]]$ and $r \in \mathbb{N}$ such that

$$a_1 s_1 + \cdots + a_n s_n = t^r.$$

If we write $s_i = \sum_j s_{i,j}t^j$ we can conclude that $(s_{i,j} \mid j \leq r) = R$. For the finite set S of nonzero $s_{i,j}$ with $j \leq r$, we let $V_s = \operatorname{Spec} R_s$, then define $V_J^\circ = \operatorname{Spec} R_J$ for each subset $J \subset S$. We can give scheme structures to V_J°'s such that the map $\coprod_J V_J^\circ \longrightarrow \operatorname{Spec} R$ is a ubi covering and for each i, j and J, the element $s_{i,j}$ is either 0 or invertible in R_J. For each J, there exists an index i_0 such that $s_{i_0} = t^q\omega$ with $\omega \in R_J[[t]]^*$. In particular $s_{i_0} \in R_J((t))^*$. This implies that the map $\operatorname{Spec} R_J((t)) \longrightarrow \operatorname{Spec} R((t))$ factors through U_{i_0}. □

7. Uniformization

If K is an algebraically closed field, then any finite étale $K((t))$-algebra A is K-isomorphic to a product of the power series field, $K((u))^n$, for some $n \in \mathbb{N}$, and its integer ring $\mathcal{O}_A$ is isomorphic to $K[[u]]^n$. This is no longer true if we replace K with a general ring. The goal of this section is to show that this however becomes true after taking a sur covering of $\operatorname{Spec} K$.

Definition 7.1. Let R be a ring and A be a finite étale $R((t))$-algebra. We say that A is *uniformizable* (over R) if there exist a finite decomposition $R \cong \prod_{i=1}^{l} R_i$, $n_i \in \mathbb{N}$ and isomorphisms $A \otimes_R R_i (= A \hat{\otimes}_R R_i) \cong R_i((s))^{n_i}$ such that each composition $R_i((t)) \longrightarrow A \otimes_R R_i \longrightarrow R_i((s))$ maps t to a series of the form $s^k g$ for some $k > 0$ and $g \in R_i[[s]]^*$.

Remark 7.2. If $A/R((t))$ is an uniformizable finite étale $R((t))$-algebra and we use notation from 7.1 then $\mathcal{O} = \prod_i R_i[[s]]^{n_i}$ is a finite and flat $R[[t]]$ algebra with an isomorphism $\mathcal{O}_t \cong A$. It is not clear if a general finite étale $R((t))$-algebra always admits a finite and flat extension on $R[[t]]$, not even fpqc locally.

Theorem 7.3 (Uniformization)**.** *Let R be a ring and A be a finite and étale $R((t))$-algebra. Then there exists a surjective and finitely presented map $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ such that $A \hat{\otimes}_R S$ is uniformizable. In other words A is uniformizable sur locally in R.*

Proof. Let S_0 be the henselization of $R[t]$ with respect to the ideal (t). From [4, Th. 7 and pages 588-589] or [6, Th. 5.4.53], there exists a finite étale cover $S_0[t^{-1}] \longrightarrow A_0$ such that $A_0 \otimes_{S_0[t^{-1}]} R((t)) \cong A$. In turn there exist an étale neighborhood $R[t] \longrightarrow S_1$ of (t), that is with $R \cong S_1/tS_1$, and a finite étale cover $S_1[t^{-1}] \longrightarrow A_1$ such that $A_1 \otimes_{S_1[t^{-1}]} S_0[t^{-1}] \cong A_0$ and $A_1 \otimes_{S_1[t^{-1}]} R((t)) \cong A$. Since S_1 and A_1 are finitely generated over R, there exist a finitely generated subalgebra $R' \subset R$, an étale neighborhood $R'[t] \longrightarrow S_2$ and a finite étale cover $S_2[t^{-1}] \longrightarrow A_2$ which induce $R[t] \longrightarrow S_1$ and $S_1[t^{-1}] \longrightarrow A_1$ by the scalar extension R/R'. Then $A \cong A_2 \otimes_{S_2[t^{-1}]} R((t))$. If we put $A' = A_2 \otimes_{S_2[t^{-1}]} R'((t))$, then $A \cong A' \otimes_{R'((t))} R((t))$. Therefore it suffices to show that $R'((t)) \longrightarrow A'$ is sur locally uniformizable.

We claim that there exist a sur covering $\coprod_i \operatorname{Spec} R_i \longrightarrow \operatorname{Spec} R'$ and a commutative diagram for each i,

$$\begin{array}{ccc} A_2 \otimes_{R'} R_i & \longrightarrow & \operatorname{Spec} Q_i \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} S_2 \otimes_{R'} R_i & \end{array}$$

such that

1. R_i is a domain,
2. we have $(Q_i)_t \cong A_2 \otimes_{R'} R_i$,
3. the lower left arrow is the one induced from $S_2 \longrightarrow S_2[t^{-1}] \longrightarrow A_2$,
4. the lower right arrow is a finite morphism,
5. each connected component of $(\operatorname{Spec} Q_i/tQ_i)_{\text{red}}$ maps isomorphically onto $\operatorname{Spec} R_i = \operatorname{Spec} S_2 \otimes_{R'} R_i/t(S_2 \otimes_{R'} R_i)$,
6. up to shrink S_2 to a smaller neighborhood of (t), $\sqrt{tQ_i}$ is a principal ideal generated by some $q_i \in Q_i$.

Let's see how to conclude from this. By [3, Corollary 7.5], the ring $Q_i \otimes R_i[[t]]$ is a product of rings $P_1 \times \cdots \times P_l$ such that the reduction of P_j/tP_j is R_i. The map $R[[x]] \longrightarrow P_i$, $x \longrightarrow q_i$ is well defined and surjective because P_i is t-adically and therefore q_i-adically complete. Since $\dim R[[x]] = \dim P_i$ and $R[[x]]$ is a domain the map $R[[x]] \longrightarrow P_i$ is an isomorphism. In conclusion $Q_i \otimes R_i[[t]] \cong R_i[[q_i]]^{n_i}$ for some $n_i \in \mathbb{N}$.

This implies that $A_2 \otimes_{S_2[t^{-1}]} R_i((t)) \cong A' \otimes_{R'((t))} R_i((t)) \cong R_i((q_i))^{n_i}$. The image of t in each factor $R_i[[q_i]]$ is of the form $q_i^k g$ for some $k > 0$ and $g \in R_i[[q_i]] \backslash q_i R_i[[q_i]]$.

Inverting the constant term of g for each factor, we get an open dense subscheme $\operatorname{Spec} R'_i \subset \operatorname{Spec} R_i$ such that $R'_i((t)) \longrightarrow A' \otimes_{R'((t))} R'_i((t))$ is uniformizable. By Noetherian induction, we conclude that $R_i((t)) \longrightarrow A' \otimes_{R'((t))} R_i((t))$ is sur locally uniformizable. Therefore $R'((t)) \longrightarrow A'$ is also sur locally uniformizable and the theorem follows.

It remains to prove the claim. Note that R' is finitely generated over $\mathbb{Z}$, in particular, a Noetherian ring of finite dimension. By 4.29, we may assume that R' is a domain and it is enough to show that there exists one dominant finite-type morphism $\operatorname{Spec} R_i \longrightarrow \operatorname{Spec} R'$ satisfying the above conditions. Let K be an algebraic closure of the fraction field K' of R'. The map $\operatorname{Spec} A_2 \otimes_{R'} K \longrightarrow \operatorname{Spec} S_2[t^{-1}] \otimes_{R'} K$ is an étale finite cover of affine algebraic curves over K. Taking a partial compactification of $\operatorname{Spec} A_2 \otimes_{R'} K$, we can extend this cover to a finite (not necessarily étale) cover $\operatorname{Spec} Q_K \longrightarrow \operatorname{Spec} S_2 \otimes_{R'} K$ with $\operatorname{Spec} Q_K$ smooth. Let $p_1, \dots, p_m \colon \operatorname{Spec} K \longrightarrow \operatorname{Spec} Q_K$ be the points lying over the point $\operatorname{Spec} K = V(t) \hookrightarrow \operatorname{Spec} S_2 \otimes_{R'} K$. We take a sufficiently large intermediate field L between K and K' which is finite over K' and such that Q_K and morphisms $\operatorname{Spec} Q_K \longrightarrow \operatorname{Spec} S_2 \otimes_{R'} K$ and p_i are all defined over L (see [9, Tag 01ZM], [9, Tag 01ZN]). Denote by $Q_L \longrightarrow \operatorname{Spec} S_2 \otimes_{R'} L$ the obtained map. As $Q_L \otimes_L K \cong Q_K$ we can conclude that Q_L is smooth over L. Replacing R' with its integral closure in L, we can assume $K' = L$. Since $\operatorname{Spec} Q_{K'}$ is normal, we can extend $\operatorname{Spec} Q_{K'} \longrightarrow \operatorname{Spec}(S_2 \otimes_{R'} K')$ to $\operatorname{Spec} S_2$ by taking the normalization $\operatorname{Spec} Q \longrightarrow \operatorname{Spec} S_2$. As $\operatorname{Spec} Q_t \longrightarrow \operatorname{Spec} S_2[t^{-1}]$ and $\operatorname{Spec} A_2 \longrightarrow \operatorname{Spec} S_2[t^{-1}]$ are isomorphic over K, shrinking $\operatorname{Spec} R'$ we can assume that $Q_t \cong A_2$ (see [9, Tag 081E]). So far we have proved that $\operatorname{Spec} Q$ satisfies points (1) to (4).

For (5), consider $\operatorname{Spec}(Q/tQ)_{\mathrm{red}} \longrightarrow \operatorname{Spec}(S_2/tS_2) = \operatorname{Spec} R'$. We have

$$(Q/tQ)_{\mathrm{red}} \otimes_{R'} K' \cong (Q_{K'}/tQ_{K'})_{\mathrm{red}} \cong K'^m$$

because $(Q_{K'}/tQ_{K'})$ is a finite K' algebra with only rational points. Thus, shrinking R', we can assume $(Q/tQ)_{\mathrm{red}} \cong R'^m$, that is point (5).

We now focus on (6). Since $\operatorname{Spec} Q \longrightarrow \operatorname{Spec} R'$ is generically smooth, by [9, Tag 0C0C] we can assume it is smooth. Since R' is excellent, we can also assume that R' is regular, so that Q is regular as well. Let $J = \sqrt{tQ}$. If $x = tS_2 \in \operatorname{Spec} S_2$ is the unique point over $0 = (t) \in \operatorname{Spec} R'[t]$, we have that that Q_x is a finite extension of $(S_2)_x \cong R[t]_{(t)}$. In particular $Q_x \otimes_{R'} K'$ is a semilocal regular ring of dimension 1, thus a finite product of semilocal Dedekind domains. By [1, Chapter 13, Corollary 1.4] it is therefore a PID, so that $(J_x) \otimes_{R'} K' \cong Q_x \otimes_{R'} K'$. By [9, Tag 01ZM] we can assume it extends to an isomorphism $J_x \cong Q_x$. Shrinking S_2 étale locally around x, we can finally assume $J \cong Q$, which yields condition (6). □

8. The P-moduli space of formal torsors

Let k be a base field and G be a finite group. We prove the existence of P-moduli space of torsors over $k((t))$ for a fixed finite group G or the one of finite étale covers of $k((t))$ of fixed degree.

Notation 8.1. We set $p = \operatorname{char} k$, allowing it also to be 0. In this section, with abuse of notation, we often consider groups of the form $H \rtimes C$ where H is a p-group and C is a tame cyclic group. If $p > 0$ this means that C is a cyclic group whose order is coprime with p. If $p = 0$ instead this means that $H = 0$, while C is any cyclic group.

Definition 8.2. The functor $\Delta_n\colon (\mathbf{Aff}/k)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ maps a ring R to the set of isomorphism classes of finite étale covers of $R((t))$ of constant degree n. For a morphism $f\colon \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ of affine k-schemes the *pull-back* map $f^*: \Delta_n(R) \longrightarrow \Delta_n(S)$ sends an étale $R((t))$-algebra A to $A\hat{\otimes}_R S = A \otimes_{R((t))} S((t))$.

We define a functor $\Delta_G\colon (\mathbf{Aff}/k)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ mapping a ring R to the set of isomorphism classes of G-tosors over $R((t))$. The pullback is defined similarly to the one of Δ_n.

Since finite étale algebras correspond to S_n-torsors by [12, Prop. 1.6], we have an isomorphism $\Delta_n \cong \Delta_{S_n}$.

Remark 8.3. Notation above slightly differs to the notation used in [10], where Δ_G denotes the analogous fiber category.

Lemma 8.4. *Suppose that G is the semidirect product $H \rtimes C$ of a p-group H and a tame cyclic group C. Then the functor Δ_G has a strong P-moduli space which is the disjoint union of countably many affine schemes of finite type over k.*

Proof. This follows from [10, Theorem A] and Lemma 4.30. $\square$

Definition 8.5. For $\mathcal{F}\colon (\mathbf{Aff}/k)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ and $A \in \mathcal{F}(V)$, a *geometric fiber* of A is the image of A under the map $\mathcal{F}(V) \longrightarrow \mathcal{F}(K)$ associated with some geometric point $\operatorname{Spec} K \longrightarrow V$.

Definition 8.6. We denote by Δ_n° (resp. Δ_G°) the subfunctor of Δ_n (resp. Δ_G) consisting of étale $R((t))$-algebras A whose geometric fibers are connected: for every algebraically closed R-field K, the induced algebra $A\hat{\otimes}_R K$ is a field.

Lemma 8.7. *The property "being connected" is locally constructible for both Δ_G and Δ_n.*

Proof. Let $\mathcal{F} = \Delta_G$ or $\mathcal{F} = \Delta_n$ and let $\mathcal{Q} \subseteq |\mathcal{F}|$ denote the property in the statement. Given $\psi\colon V \longrightarrow \mathcal{F}$ we have to show that $|\psi|^{-1}(\mathcal{Q}) = \mathcal{Q}_V \subseteq V$ is locally constructible. By 4.23 the problem is sur local in V. By Theorem 7.3 we can therefore assume that $V = \operatorname{Spec} R$ and that ψ corresponds to a torsor/étale map $R((t)) \longrightarrow A = R((s))^m$ with $t = s^k g$, $g \in R[[s]]^*$ and $k, m > 0$. The subset $\mathcal{Q}_V$ is locally constructible in V because, if $m = 1$ then $\mathcal{Q}_V = V$, while if $m > 1$ then $\mathcal{Q}_V = \emptyset$. $\square$

Lemma 8.8. *Let $f\colon X \longrightarrow S$ be a G-torsor over a scheme S and let $X = \coprod_i U_i$ be a finite decomposition into open subsets. Then:*

- *there exist finite decompositions into open subsets $S = \coprod_j S_j$ and $f^{-1}(S_j) = \coprod_k V_{jk}$ with the following properties: for all j, k there exists i such that $V_{jk} \subseteq U_i$; the group G permutes the V_{jk} and, for all j, G acts transitively on $\{V_{jk}\}_k$;*
- *if G permutes transitively the U_i and G_i is the stabilizer of U_i in G then U_i is an G_i-torsor over S.*

Proof. Let $T = \{g(U_i)\}_{i,g\in G}$ and, for $J \subseteq T$, set

$$T_J = (\bigcap_{V\in J} V) \cap (\bigcap_{V\notin J} (X - V))$$

so that X is the disjoint union of the T_J and all U_i are a disjoint union of some of the T_J. Notice that G permutes the T_J: given $g \in G$ and $J \subseteq T$ one has that $g(T_J) =$

$T_{g(J)}$ where $g(J) = \{g(V) \mid V \in J\}$. This also implies that $f(T_J) \cap f(T_{J'}) \neq \emptyset$ only if there exists $g \in G$ such that $g(T_J) = T_{J'}$, in which case $f(T_J) = f(T_{J'})$: if $s = f(x) = f(y)$ with $x \in T_J$ and $y \in T_{J'}$ then there exists $g \in G$ such that $g(x) = y \in g(T_J) \cap T_{J'}$, so that $g(T_J) = T_{J'}$. Up to removing repetitions the sets $f(T_J)$ yield the desired decomposition of S.

We now consider the last statement. If $S' \longrightarrow S$ is any map with $S' \neq \emptyset$ denote by $\psi\colon X' \longrightarrow X$ its base change along $f\colon X \longrightarrow S$. The collection $\{\psi^{-1}(U_i)\}_i$ defines a partition of X' over which G acts transitively. In particular $\psi^{-1}(U_i) \neq \emptyset$ for all i. We claim that G_i is also the stabilizer of $\psi^{-1}(U_i)$ in G. Indeed if $g(\psi^{-1}(U_i)) = \psi^{-1}(g(U_i)) = \psi^{-1}(U_i)$ then $g(U_i) \cap U_i \neq \emptyset$ and therefore $g(U_i) = U_i$. In particular we can assume $X = G \times S$ and, by transitivity, that $\{1\} \times S \in U_i$, so that $G_i \times S \subseteq U_i$. In order to prove that this last inclusion is an equality, it suffices to show the corresponding equality over each point of S and thus we can further assume $S = \operatorname{Spec} K$, for some field K. In this case

$$g \in U_i \implies g = g \cdot 1 \in g(U_i) \cap U_i \neq \emptyset \implies g(U_i) = U_i \implies g \in G_i$$

□

Theorem 8.9. *Let k be a field, G a finite group and let $\mathcal{Q}$ be a locally constructible property for Δ_G. Then $\Delta_G^{\mathcal{Q}}$ (e.g. Δ_G or Δ_G°) has a strong P-moduli space which is a countable disjoint union of affine k-varieties.*

Proof. By 4.24 and 4.29 it is enough to consider the case of Δ_G.

In the semidirect case $G = H \rtimes C$, for a p-group H and a tame cyclic group C, the claim follows from 8.4. In this case we denote the P-moduli space of Δ_G° by $\overline{\Delta_G^{\circ}}$.

Let now G be an arbitrary finite group. Let Λ be the set of representatives of G-conjugacy classes of subgroups $H \subset G$ which are isomorphic to the semi-direct product $B \rtimes C$ of a p-group B and a tame cyclic group C. Let also $\operatorname{Aut}_G(H)$ denote the subgroup of automorphisms of H induced by conjugation of an element of G. There exists a natural action of $\operatorname{Aut}_G(H)$ on Δ_H°, inducing a natural P-action on $\overline{\Delta_H^{\circ}}$. From 4.44, there exists the strong P-quotient $\overline{\Delta_H^{\circ}}/\operatorname{Aut}_G(H)$, which is a strong P-moduli space of the quotient functor $\Delta_H^{\circ}/\operatorname{Aut}_G(H)$ and it is P-isomorphic to a disjoint union of k-varieties.

Consider the map $\operatorname{ind}_H^G\colon \Delta_H^{\circ} \longrightarrow \Delta_G$. This is $\operatorname{Aut}_G(H)$-invariant and induces maps $\Delta_H^{\circ}/\operatorname{Aut}_G(H) \longrightarrow \Delta_G$ and

$$\coprod_{H \in \Lambda} \Delta_H^{\circ}/\operatorname{Aut}_G(H) \longrightarrow \Delta_G.$$

We claim that this is geometrically injective and an epimorphism in the sur topology. Since the source of this map has a strong P-moduli space as in the theorem, from 4.14, the claim implies the theorem. It remains to show the claim.

Epimorphism. Follows from 7.3 and 8.8 and the fact that Galois extensions of $K((t))$ with K algebraically closed has Galois group a semidirect product of a p-group and a cyclic tame group.

Geometrically injective. If K is an algebraically closed field then $\Delta_H^{\circ}(K)/\operatorname{Aut}_G(H)$ is the set of isomorphism classes of Galois extensions $L/K((t))$ modded out by the equivalence relation induced by the action of $\operatorname{Aut}_G(H)$. Given such an object the corresponding G-torsor is $\operatorname{ind}_H^G L$. Let $L \in \Delta_H^{\circ}(K)$ and $L' \in \Delta_{H'}^{\circ}(K)$ for $H, H' \in \Lambda$

be such that $\mathrm{ind}_H^G L \cong \mathrm{ind}_{H'}^G L'$ as G-torsors. It follows that L' is one of the component of $\mathrm{ind}_H^G L$ and H' is its stabilizer. Thus $H' = gHg^{-1}$ for $g \in G$ and $L' = L$ with the H' action induced by H. But since Λ is a set of representative we obtain $H' = H$ and therefore $L, L' \in \Delta_H^\circ(K)$ are in the same orbit for the action of $\mathrm{Aut}_G(H)$. □

Corollary 8.10. *Theorem 8.9 holds also when G is a finite étale group scheme over k.*

Proof. Let k'/k be a finite Galois extension with Galois group H such that $G \otimes_k k'$ is a constant group. After the base change to k', the functor Δ_G^Q has a strong P-moduli space. This space has a P-action of H. It suffices to take the strong P-quotient, which exists from 4.44. □

9. Local constructibility of weighting functions

In the wild McKay correspondence, there appear motivic integrals of the form $\int_{\Delta_G} \mathbb{L}^f$ for some weighting functions $f \colon \Delta_G \longrightarrow \frac{1}{|G|}\mathbb{Z}$. In this section, we show that these functions f are locally constructible, which proves that these integrals indeed make sense.

We first recall the definitions of these functions. We fix a free $k[[t]]$-module $M = k[[t]]^{\oplus r}$ of rank r endowed with a $k[[t]]$-linear G-action. For a k-algebra B, we let $M_B := M \hat{\otimes}_k B$.

Definition 9.1. For a field extension K/k and a G-torsor $A/K((t))$, we define a number $v_M(A) \in \frac{1}{|G|}\mathbb{Z}$ by

$$v_M(A) := \frac{1}{|G|} l_{K[[t]]} \left(\frac{\mathrm{Hom}_{k[[t]]}(M, \mathcal{O}_A)}{\mathcal{O}_A \cdot \mathrm{Hom}_{k[[t]]}^G(M, \mathcal{O}_A)} \right),$$

where $\mathrm{Hom}_{k[[t]]}^G(M, \mathcal{O}_A)$ is the set of G-equivariant $k[[t]]$-linear maps, $\mathcal{O}_A$ is the integral closure of $K[[t]]$ inside A and $l_{K[[t]]}$ denotes the length of a $K[[t]]$-module. The natural map

$$M_K \longrightarrow E_K := \mathrm{Hom}_{K[[t]]}(\mathrm{Hom}_{K[[t]]}^G(M_K, \mathcal{O}_A), \mathcal{O}_A),\ f \longmapsto (\psi \longmapsto \psi(f))$$

induces a map

$$\eta_A \colon \mathrm{Spec}\, S^\bullet_{\mathcal{O}_A} E_K \longrightarrow \mathrm{Spec}\, S^\bullet_{K[[t]]} M_K,$$

where $S^\bullet_R M$ denotes the symmetric algebra of an R-module M. Let

$$o \in (\mathrm{Spec}\, S^\bullet_{K[[t]]} M_K)(K)$$

be the K-point at the origin. We define

$$w_M(A) := \dim \eta_A^{-1}(o) - v_M(A) \in \frac{1}{|G|}\mathbb{Z}.$$

Using the above functions we define maps

$$v_M, w_M \colon \Delta_G \longrightarrow \frac{1}{|G|}\mathbb{Z}.$$

We will just write $v = v_M$ and $w = w_M$ when this creates no confusion.

Remark 9.2. For slight differences of definitions of v and w appearing in the literature, see [17, Rem. 8.2] and errata to the paper [14], available online.[1]

[1] https://msp.org/ant/2017/11-4/p02.xhtml

We are going to prove that the maps $v, w\colon \Delta_G \longrightarrow \frac{1}{|G|}\mathbb{Z}$ are well defined and locally constructible. If we set $N = \mathrm{Hom}_{k[[t]]}(M, k[[t]])$ and think of it as a $k[[t]]$-module with an action of G we have isomorphisms

$$N \otimes_{k[[t]]} \mathcal{O}_A \cong \mathrm{Hom}_{k[[t]]}(M, \mathcal{O}_A) \cong N_K \otimes_{K[[t]]} \mathcal{O}_A \cong \mathrm{Hom}_{K[[t]]}(M_K, \mathcal{O}_A).$$

Lemma 9.3. *The $K[[t]]$-module $\mathrm{Hom}^G_{k[[t]]}(M, \mathcal{O}_A)$ is free of rank r and the map*

$$\mathrm{Hom}^G_{k[[t]]}(M, \mathcal{O}_A) \otimes_{K[[t]]} \mathcal{O}_A \longrightarrow \mathcal{O}_A \cdot \mathrm{Hom}^G_{k[[t]]}(M, \mathcal{O}_A) \subseteq \mathrm{Hom}_{k[[t]]}(M, \mathcal{O}_A)$$

is an isomorphism. In particular the number $v(A)$ is well defined, that is finite.

Proof. We can assume $K = k$. The module $\mathrm{Hom}^G_{k[[t]]}(M, \mathcal{O}_A)$ is contained in a free $k[[t]]$-module and therefore it is free. In order to compute its rank and prove that the map in the statement is injective we can check what happens after localizing by t. If we set $R = k((t))$ and $N_t = Q$ we have that the map $(Q \otimes_R A)^G \otimes_R A \longrightarrow Q \otimes_R A$, fppf locally on R after trivializing A, become

$$(Q \otimes_R R[G])^G \otimes_R R[G] \longrightarrow Q \otimes_R R[G].$$

In particular $(Q \otimes_R R[G])^G \cong Q$, the corresponding map $Q \longrightarrow Q \otimes_R R[G]$ is the coaction and the above map is an isomorphism. □

Remark 9.4. The function v is equal to the t-order of the ideal $\mathfrak{r}^*(L|\mathfrak{o}, M) \subset \mathcal{O}_A$, Fröhlich's module resolvent [5, Sec. 3]. This follows from determinantal descriptions of both values (see [16, Def. 6.5], [15, Def. 3.3] and [5, Sec. 3]).

Lemma 9.5. *Let $K'/K/k$ be field extensions, $A/K((t))$ a G-torsor and $A_{K'} = A\hat{\otimes}_K K'$ the associated G-torsor over $K'((t))$. Then $v(A) = v(A_{K'})$ and $w(A) = w(A_{K'})$. In particular the maps $v, w\colon \Delta_G \longrightarrow \frac{1}{|G|}\mathbb{Z}$ are well defined.*

Proof. Recall that the operations of taking invariants and flat base change commute. From 9.3 and the fact that

$$l_{K'[[t]]}(Q\hat{\otimes}_K K') = l_{K[[t]]}(Q) \text{ for } Q \in \mathrm{Mod}(K[[t]])$$

one gets $v(A) = v(A_{K'})$. Similarly one obtains that $\eta^{-1}_{A_{K'}}(o) \cong \eta^{-1}_A(o) \times_K K'$ and therefore the equality for the dimensions. □

Lemma 9.6 (cf. [15, Lem. 3.4])**.** *Let H be a subgroup of G. Then we have the equalities of functions*

$$v_{\mathrm{Res}_H M} = v_M \circ \mathrm{ind}^G_H, w_{\mathrm{Res}_H M} = w_M \circ \mathrm{ind}^G_H\colon \Delta_H \longrightarrow \Delta_G \longrightarrow \frac{1}{|G|}\mathbb{Z}.$$

Proof. Assume a G-torsor $A/K((t))$ is induced by an H-torsor $B/K((t))$. We have isomorphisms

$$A \cong \mathrm{ind}^G_H B \cong B^{|G/H|},\ \mathcal{O}_A \cong \mathrm{ind}^G_H \mathcal{O}_B \cong \mathcal{O}_B^{|G/H|}$$

and $\mathrm{Hom}^G_{k[[t]]}(M, \mathcal{O}_A) \cong \mathrm{Hom}^H_{k[[t]]}(M, \mathcal{O}_B)$. Moreover

$$\mathcal{O}_A \mathrm{Hom}^G_{k[[t]]}(M, \mathcal{O}_A) \cong (\mathcal{O}_B \mathrm{Hom}^H_{k[[t]]}(M, \mathcal{O}_B))^{|G/H|}$$

inside $(\mathrm{Hom}_{k[[t]]}(M, \mathcal{O}_B))^{|G/H|} \cong \mathrm{Hom}_{k[[t]]}(M, \mathcal{O}_A)$. This proves that $v(A) = v(B)$. Finally one can check that

$$\eta_A\colon \mathbb{A}^r_{\mathcal{O}_A} = \mathbb{A}^r_{\mathcal{O}_B} \amalg \cdots \amalg \mathbb{A}^r_{\mathcal{O}_B} \longrightarrow \mathbb{A}^r_{K[[t]]}$$

and that all maps $\mathbb{A}^r_{\mathcal{O}_B} \longrightarrow \mathbb{A}^r_{K[[t]]}$ are isomorphic to η_B. It follows that $\dim \eta_A^{-1}(o) = \dim \eta_B^{-1}(o)$. □

To show properties of v and w, we give slightly different descriptions of these functions. For simplicity assume that A is uniformizable and connected, that is a Galois extension of $K((t))$ with group G and $\mathcal{O}_A = K[[s]]$. Let

$$a_j = {}^t(a_{1j}, \dots, a_{rj}) \in N \otimes_{k[[t]]} \mathcal{O}_A \cong \mathcal{O}_A^r \quad (j = 1, \dots, r)$$

be a $K[[t]]$-basis of $(N \otimes_{k[[t]]} \mathcal{O}_A)^G$, which, by 9.3, is also an $\mathcal{O}_A$-basis of $\mathcal{O}_A(N \otimes_{k[[t]]} \mathcal{O}_A)^G$. Since $l_{K[[t]]} = l_{K[[s]]}$ and using standard properties of DVR's we get

$$v(A) = \frac{\operatorname{ord}_A \det(a_{ij})}{|G|}, \tag{9.1}$$

where ord_A denotes the normalized additive valuation on A, that is, the order in s.

Let $e_1, \dots, e_r$ be the standard basis of M and $b_1, \dots, b_r \in E_K$ the dual basis of $a_1, \dots, a_r$. The map $M_K \longrightarrow E_K$ sends e_i to $\sum_j a_{ij} b_j$. If Ω is the residue field of $\mathcal{O}_A$ then

$$(\eta_A^{-1}(o))_{\mathrm{red}} = \operatorname{Spec}\left(\frac{\Omega[X, \dots, X_r]}{(\sum_j \overline{a_{ij}} X_j)}\right)$$

where $\overline{a_{ij}}$ is the image of $a_{ij} \in \mathcal{O}_A$ in Ω. It follows that

$$\dim \eta_A^{-1}(o) = r - \operatorname{rank}(\overline{a_{ij}}). \tag{9.2}$$

Lemma 9.7. *Let $A/B((t))$ be a G-torsor such that $A = B((s))$ is uniformizable and $\mathcal{O}_A = B[[s]]$ is G invariant. Assume moreover that B is a Noetherian ring. Then there exist a sur covering $\operatorname{Spec} B' \longrightarrow \operatorname{Spec} B$ such that $(N \otimes_{k[[t]]} (\mathcal{O}_A \hat{\otimes}_B B'))^G$ is a free $B'[[t]]$-module of rank r and, for any ring map $B' \longrightarrow C$, the base change map*

$$[N \otimes_{k[[t]]} (\mathcal{O}_A \hat{\otimes}_B B')]^G \hat{\otimes}_{B'} C \longrightarrow [N \otimes_{k[[t]]} (\mathcal{O}_A \hat{\otimes}_B C)]^G$$

is an isomorphism.

Proof. By 4.28 we may suppose that B is a domain and show that there exists an affine open dense subscheme $\operatorname{Spec} B' \hookrightarrow \operatorname{Spec} B$ satisfying the requests of the lemma. Given a B-algebra C we set $A_C = A \hat{\otimes}_B C$, $\mathcal{O}_{A_C} = \mathcal{O}_A \hat{\otimes}_B C \cong C[[s]]$ and

$$\phi_{A_C} \colon N \otimes_{k[[t]]} \mathcal{O}_{A_C} \longrightarrow \bigoplus_{g \in G} N \otimes_{k[[t]]} \mathcal{O}_{A_C}, \alpha \longmapsto (g\alpha - \alpha)_g$$

so that $(N \otimes_{k[[t]]} \mathcal{O}_{A_C})^G = \operatorname{Ker} \phi_{A_C}$. Let S be the localization of $B[[t]]$ at the prime ideal (t), which is a discrete valuation ring. Let us consider the map

$$\phi_A \otimes_{B[[t]]} S \colon N \otimes_{k[[t]]} \mathcal{O}_A \otimes_{B[[t]]} S \longrightarrow \bigoplus_{g \in G} N \otimes_{k[[t]]} \mathcal{O}_A \otimes_{B[[t]]} S.$$

From [2, VII. 21], there exist S-bases $\alpha_1, \dots, \alpha_e$ and $\beta_1, \dots, \beta_f$ of the source and the target, and elements $c_1, \dots, c_e \in S$ such that $\phi \otimes_{B[[t]]} S$ sends α_i to $c_i \beta_i$. Moreover, we may suppose that for some $d \in \{1, \dots, e\}$, we have $c_i = 0$, $i \le d$ and $c_i \neq 0$, $i > d$.

Identifications $N \cong k[[t]]^{\oplus r}$ and $\mathcal{O}_A \cong B[[t]]^{|G|}$ induce an identification

$$N \otimes_{k[[t]]} \mathcal{O}_A \otimes_{B[[t]]} S \cong S^{r|G|}.$$

Through this identification, α_i and β_i are expressed as tuples $(\alpha_{i,j})_j$ and $(\beta_{i,j})_j$ of elements of S. Note that an element of S is a fraction u/v with $u, v \in B[[t]]$ such that v has nonzero constant term, denoted by v_0, and v is invertible in the ring $B_{v_0}[[t]]$. In particular there exists $v \in B[[t]] - (t)$ such that the S-bases α_i and β_j are also bases over $B[[t]]_v$. Replacing B by B_{v_0} we can therefore assume that this holds globally. In particular $c_i \in B[[t]]$ and we may further suppose that the leading coefficients of c_i, $i > d$ are units, that is they are invertible. Then, for any ring map $B \longrightarrow C$, the map $\phi_{A_C} = \phi_A \otimes_{B[[t]]} C[[t]]$ is similarly given by $\alpha_i \longmapsto c_i\beta_i$, where $c_i \in C[[t]]$ are zero for $i \leq d$ and units for $i > d$. This ends the proof. □

Theorem 9.8. *The functions $v, w \colon \Delta_G \longrightarrow \frac{1}{|G|}\mathbb{Z}$ are locally constructible (see 4.25). Similarly for the restrictions of v and w to $\Delta_G^{\mathcal{Q}}$ for a locally constructible property $\mathcal{Q}$.*

Proof. The second assertion is a direct consequence of the first by 8.9 and 4.26. We are going to prove the first assertion, that is prove that, if $Z \longrightarrow \Delta_G$ is a map from a scheme, then the restrictions $v_{|Z}$ and $w_{|Z}$ are locally constructible. By 4.26 and 7.3 we can assume that $Z = \operatorname{Spec} B$ and that the associated G-torsor $A/B((t))$ is uniformizable. We can further assume that B is a domain and, by 6.8, 8.8 and 9.6 we may suppose that $A = B((s))$ is also a domain and $\mathcal{O}_A = B[[s]]$ is G-invariant. In particular we can now apply 9.7 and assume the conclusion of this lemma. Let $a_1, \dots, a_r$ be a $B[[t]]$-basis of $(N \otimes_{k[[t]]} \mathcal{O}_A)^G$ and write

$$a_j = (a_{ij})_i \in (N \otimes_{k[[t]]} \mathcal{O}_A) \cong \mathcal{O}_A^r.$$

We want to use equation (9.1). Write

$$d = \det(a_{ij}) \in \mathcal{O}_A = B[[s]]$$

as $\sum_{i\geq 0} d_i s^i$, $d_i \in B$. Then the locus where this determinant has s-order $\geq l$ is the closed subset

$$\{v \geq l/|G|\} = \{p \in \operatorname{Spec} B \mid \operatorname{ord}_{A_{k(p)}}(d) \geq l\} = \bigcap_{i<l} V(d_i) \subset \operatorname{Spec} B.$$

As for the function w, let $\overline{a_{ij}}$ be the image of a_{ij} in B and consider the matrix $(\overline{a_{ij}}) \in B^{r^2}$. From equation (9.2), we need to show that the map

$$\operatorname{Spec} B \ni x \longmapsto \operatorname{rank}(\overline{a_{ij}})_x$$

is locally constructible. The locus where this rank is less than s is the zero locus of the $s \times s$ minors of $(\overline{a_{ij}})$ and is a closed subset. This completes the proof. □

Corollary 9.9. *Let l be a positive integer such that $v(\Delta_G) \subset \frac{1}{l}\mathbb{Z}$ (e.g. $l = |G|$). Integrals $\int_{\Delta_G} \mathbb{L}^{d-v}$ and $\int_{\Delta_G} \mathbb{L}^{w}$ are well-defined as elements of $\hat{\mathcal{M}}_k^{\mathrm{mod},l} \cup \{\infty\}$.*

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